ON WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR THE WAVE EQUATION IN STATIC SPHERICALLY SYMMETRIC SPACETIMES

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Abstract. We give simple conditions implying the well-posedness of the Cauchy problem for the propagation of classical scalar fields in general $(n+2)$-dimensional static and spherically symmetric spacetimes. They are related to properties of the underlying spatial part of the wave operator, one of which being the standard essentially self-adjointness. However, in many examples the spatial part of the wave operator turns out to be not essentially selfadjoint, but it does satisfy a weaker property that we call here *quasi essentially self-adjointness*, which is enough to ensure the desired well-posedness. This is why we also characterize this second property.

We state abstract results, then general results for a class of operators encompassing many examples in the literature, and we finish with the explicit analysis of some of them.

1. Introduction

This paper is a continuation of a previous one [1], tackling the well posedness of Cauchy problem for waves in static spacetimes.

This subject has been launched by Wald in [2], and further developed by, among others, the authors of references [3, 4, 5].

The propagation of waves is, in such spaces, described by a classical equation of the form

$$\partial_{tt}\phi + \mathcal{A}\phi = 0,$$

where $\mathcal{A}$ is a selfadjoint extension of a given symmetric and positive operator A which reflects the underlying geometry.

Our motivation relies on the following observation: although A may not be essentially selfadjoint (*e.s.a.*), boundary conditions are not necessary to construct $\mathcal{A}$ in some geometries of interest. Such a situation arises when, even if A has many selfadjoint extensions, only one has its domain included in the energy space naturally associated to A. Here we call *quasi essentially selfadjoint* (*q.e.s.a.*) this property.

We have shown in [1] that operators A given by propagation of massless scalar fields in static spacetimes with naked timelike singularities may be *q.e.s.a.* but not *e.s.a.*. Thus, in such situations, demanding the finiteness of the energy is enough to select one selfadjoint extension of A, and only one; in addition, we proved that the solutions of the wave equation may have a non trivial trace at the boundary of the geometrical domain, even though this trace is not imposed by any boundary condition at all. This phenomenon never happens with *e.s.a.* operators.

Here we deeply examine the case of general $(n+2)$-dimensional static and spherically symmetric spacetimes. More precisely, the concrete setting is the following.

The domain is of the form $I\times\mathscr{M}$, where $I\subset(0,+\infty)$ is an open interval and $\mathscr{M}$ is a compact, oriented Riemannian manifold without boundary. The operator A is defined on $C_0^\infty(I\times\mathscr{M})$ as

$$A\varphi(z,\mathbf{x}) = \frac{1}{a(z)}\left\{-\partial_z\Big(b(z)\partial_z\varphi(z,\mathbf{x})\Big) - c(z)\Delta_{\mathscr{M}}\,\varphi(z,\mathbf{x}) + d(z)\varphi(z,\mathbf{x})\right\}, \tag{1}$$

where $\Delta_{\mathscr{M}}$ is the Laplace-Beltrami operator on $\mathscr{M}$, and a, b, c, d are suitable positive coefficients only depending on the radial variable $z \in I$. No condition is prescribed on the coefficients at the boundary of the domain.

For this class of operators we fully characterize *e.s.a.* and *q.e.s.a.* properties. We then apply it to static spherically symmetric spacetimes of arbitrary dimension; in particular, we show that, in asymptotically flat spacetimes, a horizon occurs if and only if the Cauchy problem is well posed without any boundary condition; more generally, we systematically describe the situations where boundary conditions are, or are not, necessary for the Cauchy problem to be well posed.

The outline of the paper is as follows. Section 2 is devoted to abstract results on *e.s.a.* and *q.e.s.a.* properties. In section 3 we completely characterize *e.s.a.* and *q.e.s.a.* properties of the operator given in (1). We show, in section 4, the well-posedness of the Cauchy problem when the operator A is *q.e.s.a.* but not necessarily *e.s.a.*. In section 5 we apply our results to the study of propagation of scalar fields in general $(n+2)$-dimensional static and spherically symmetric spacetime with $n \geq 1$ and metric signature $(-+\cdots+)$. We close by discussing, as explicit examples in section 6, the cases of $(n+2)$-dimensional Minkowski spacetime with a removed spatial point and the higher-dimensional generalization of Schwarzschild and Reissner-Nordström geometries.

2. Quasi essentially and essentially selfadjointness

Let $\Omega \subset \mathbb{R}^{n+1}$ be a suitable domain and H a Hilbert space such that $C_0^\infty(\Omega)$ is dense in H, and let us consider a symmetric unbounded definite positive operator A defined on H with domain $D(A) = C_0^\infty(\Omega)$. We assume the existence of a bilinear symmetric form $\mathfrak{b}$ and a Hilbert space $\mathcal{E}$ having the following properties:

(1) $\mathcal{E} \subset H$ and $\|\phi\|_{\mathcal{E}}^2 = \|\phi\|_H^2 + \mathfrak{b}(\phi, \phi)$;
(2) $C_c^\infty(\Omega)$ is dense in $\mathcal{E}$, where $C_c^\infty(\Omega)$ is the space of the restrictions to Ω of $C_0^\infty(\mathbb{R}^{n+1})$;
(3) if $\phi, \psi \in C_0^\infty(\Omega)$, we have $\mathfrak{b}(\phi, \psi) = <\phi, A\psi>$.

Definition 2.1. *We shall say that $\mathcal{A}$, any given selfadjoint extension of A, is of finite energy when $D(\mathcal{A}) \subset \mathcal{E}$, with continuous injection.*

Calling $\mathcal{E}^0$ the closure of $C_0^\infty(\Omega)$ in $\mathcal{E}$, we have the following result:

Theorem 2.2. *Under these hypotheses we have:*

(1) *The operator A has only one selfadjoint extension with finite energy if and only if $\mathcal{E}^0 = \mathcal{E}$. If this is the case, this extension is A_F, the Friedrichs extension.*
(2) *If $\mathcal{E}^0 = \mathcal{E}$, then $C_0^\infty(\Omega)$ is dense in $D(A_F)$ if and only if A is essentially selfadjoint (e.s.a.), i.e., A has only one selfadjoint extension.*

Proof:

(i) To prove this assertion, we begin with assuming that A has only one selfadjoint extension with finite energy. Let $\mathcal{A}$ be the selfadjoint operator associated with the energy form $\mathfrak{b}$; let $\mathcal{A}_0$ be the selfadjoint operator associated with the restriction of $\mathfrak{b}$ to $\mathcal{E}^0$. Both are extensions of A with domains included in $\mathcal{E}$, and so, are equal. But then we must have $D(\mathcal{A}^{\frac{1}{2}}) = D(\mathcal{A}_0^{\frac{1}{2}})$, which is $\mathcal{E} = \mathcal{E}^0$.

Reciprocally, if $\mathcal{E} = \mathcal{E}^0$, the only selfadjoint extension of A with domain in $\mathcal{E}$ is its Friedrichs extension, because the form $\mathfrak{b}$ defined on $\mathcal{E}$ is the closure of the form $\mathfrak{b}$ defined on $C_0^\infty(\Omega)$.

(ii) Recall that

$$D(A^*) = \{\varphi \in H : \exists C > 0 \ : \forall \psi \in C_0^\infty(\Omega), \ | < \varphi, A\psi > | \leq C\|\psi\|_H\},$$

and that

$$D(A_F) = \{\varphi \in \mathcal{E} : \exists C > 0 \ : \forall \eta \in \mathcal{E}, \ \mathfrak{b}(\varphi, \eta) \leq C\|\eta\|_H\}.$$

We assume first that $C_0^\infty(\Omega)$ is dense in $D(A_F)$. It is enough to see that $D(A^*) \subset D(A_F)$. Taking $\phi_0 \in D(A^*)$ and $\eta_0 = (A^* + I)\phi_0$, we have for all $\psi \in C_0^\infty(\Omega)$

$$< \phi_0, (A_F + I)\psi >=< \phi_0, (A + I)\psi >=< \eta_0, \psi >$$

and then, since $C_0^\infty(\Omega)$ is dense in $D(A_F)$, for all $\varphi \in D(A_F)$

$$< \phi_0, (A_F + I)\varphi >=< \eta_0, \varphi > .$$

Taking into account that $(A_F + I)^{-1}$ is defined on all H, by calling $\varphi_0 = (A_F + I)^{-1}\eta_0 \in D(A_F)$ we have

$$< \eta_0, \varphi >=< (A_F + I)(A_F + I)^{-1}\eta_0, \varphi >=< \varphi_0, (A_F + I)\varphi > \quad \text{for all } \varphi \in D(A_F),$$

and then

$$< \varphi_0 - \phi_0, (A_F + I)\varphi >= 0 \quad \text{for all } \varphi \in D(A_F).$$

Since $\mathrm{Im}(A_F + I) = H$, we have $\varphi_0 = \phi_0$. It implies $D(A^*) \subset D(A_F)$ and so $A^* = A_F$. Then A is essentially selfadjoint.

On the other hand, if $C_0^\infty(\Omega)$ is not dense in $D(A_F)$, there exists $\varphi \in D(A_F)$ such that $A_F\varphi \neq 0$ and

$$< A_F\varphi, A_F\psi >= 0 \quad \forall\psi \in C_0^\infty(\Omega).$$

Let us call $\eta = A_F\varphi$. If $\eta \in \mathcal{E}$, then $\mathfrak{b}(\eta, \psi) =< \eta, A\psi >=< \eta, A_F\psi >= 0$ for all $\psi \in C_0^\infty(\Omega)$ and then by density of $C_0^\infty(\Omega)$ in $\mathcal{E}$, $\mathfrak{b}(\eta, \eta) = 0$. Since by hypothesis $\eta \neq 0$, we have $\eta \notin \mathcal{E}$.

Therefore, we have proved that there exists $\eta \in H$, such that $\eta \in \ker(A^*)$ but $\eta \notin \mathcal{E}$, so A cannot be essentially self adjoint. □

Definition 2.3. *Under the preceding hypotheses, the operator A is **quasi essentially selfadjoint** (q.e.s.a.) if it has only one extension with finite energy.*

Lemma 2.4. *If A is a q.e.s.a. operator, then $D(A_F) = D(A^*) \cap \mathcal{E}$.*

Proof:

Since $D(A_F) \subset D(A^*)$ by definition of A^* and $D(A_F) \subset \mathcal{E}$ by definition of A_F, then $D(A_F) \subset D(A^*) \cap \mathcal{E}$.

Conversely, let $\varphi \in D(A^*) \cap \mathcal{E}$, then

$$\mathfrak{b}(\varphi, \psi) \leq C\|\psi\|_H \quad \forall\psi \in C_0^\infty(\Omega)$$

by definition of $D(A^*)$. Since $C_0^\infty(\Omega)$ is dense in $\mathcal{E}$ and $\varphi \in \mathcal{E}$, this inequality extends to any $\psi \in \mathcal{E}$, proving that $\varphi \in D(A_F)$. □

Lemma 2.5. *If A is a q.e.s.a. operator, then the three following statements are equivalent*

(1) *A is not an e.s.a. operator.*
(2) *there exists $\varphi \in D(A^*)$ but $\varphi \notin \mathcal{E}$.*
(3) *there exists $\varphi \in D(A^*)$ non vanishing and such that $(A^* + I)\varphi = 0$.*

Proof:

(i) $\Leftrightarrow$ *(ii)*: Observe that A is an *e.s.a.* operator if and only if $A^* = A_F$, thus, by lemma 2.4, A is *e.s.a.* operator if and only if $D(A^*) \subset \mathcal{E}$.

(ii) $\Leftrightarrow$ *(iii)* Let $\varphi_0 \in D(A^*)$ and $\varphi_0 \notin \mathcal{E}$, and define $f = (A^* + I)\varphi_0 \in H$, $\varphi = (A_F + I)^{-1} f \in D(A_F)$. We have $(A_F + I)\varphi = (A^* + I)\varphi_0$ and since $\varphi \in D(A^*)$, this implies $A^*(\varphi_0 - \varphi) + (\varphi_0 - \varphi) = 0$. Finally $\varphi_0 - \varphi$ cannot identically vanish, since $\varphi_0 \notin \mathcal{E}$ while $\varphi \in \mathcal{E}$. Thus *(iii)* holds.

Conversely, let $\varphi \neq 0$ a.e., $\varphi \in D(A^*)$ such that $(A^* + I)\varphi = 0$. If $\varphi \in \mathcal{E}$, by lemma 2.4, $\varphi \in D(A_F)$, then $\varphi = 0$ a.e. since $A_F + I$ is injective, which is a contradiction. Thus, $\varphi \notin \mathcal{E}$ and *(ii)* holds. □

3. A characterization of some *q.e.s.a.* and *e.s.a.* divergence type operators

Let $\mathscr{M}$ be a Riemannian manifold of dimension n with a metric (g_{ij}). We also assume that $\mathscr{M}$ is compact, connected, without boundary and with a given orientation.

In local coordinates, for $u \in C^\infty(\mathscr{M})$ the Laplace-Beltrami operator is

$$\Delta_{\mathscr{M}} u = \operatorname{div}(\nabla_{\mathscr{M}}\, u) = \frac{\sum_{i,j=1}^n \partial_i \left(\sqrt{g}\, g^{ij} \partial_j u\right)}{\sqrt{g},}\,,$$

where g is the determinant of the metric. Let us consider in $\Omega = (0,+\infty)\times\mathscr{M}$, the operator A given by

$$A\varphi(z,\mathbf{x}) = \frac{1}{a(z)}\left\{-\partial_z\Big(b(z)\partial_z\varphi(z,\mathbf{x})\Big) - c(z)\Delta_{\mathscr{M}}\,\varphi(z,\mathbf{x}) + d(z)\varphi(z,\mathbf{x})\right\}, \tag{2}$$

for all $\varphi \in C_0^\infty(\Omega)$, where the functions a, b, c and d satisfy the following hypotheses:

- $a\,,\ c\,,\ d \in L^1_{loc}\Big((0,+\infty)\Big)$ and $b \in C\Big((0,+\infty)\Big)$,
- $a > 0$, $b > 0$, $c > 0$ and $d \geq 0$ in $(0,+\infty)$,
- a^{-1}, b^{-1}, $c^{-1} \in L^1_{loc}\Big((0,+\infty)\Big)$.

Examples will be presented in the two last sections. Let us state in advance that the coefficient d is non vanishing only in the massive case. This is why we will call *massless* the case $d = 0$.

We define the Hilbert spaces

$$H = \{\varphi \in L^2_{loc}(\Omega) : \int_\Omega |\varphi(z,\mathbf{x})|^2 a(z) d\omega_{\mathscr{M}}\, dz < \infty\},$$

and the energy space

$$\mathcal{E} = \{\varphi \in H \cap H^1_{loc}(\Omega) : \mathfrak{b}(\varphi,\varphi) < +\infty\},$$

where we denote $\omega_{\mathscr{M}}$ the natural measure in $\mathscr{M}$, and

$$\begin{aligned}\mathfrak{b}(\varphi,\psi) = \int_\Omega b(z)\, \partial_z\varphi(z,\mathbf{x})\, \overline{\partial_z\psi(z,\mathbf{x})}\, d\omega_{\mathscr{M}}\, dz\, &+ \int_\Omega c(z)\, \nabla_{\mathscr{M}}\, \varphi(z,\mathbf{x}) \cdot \overline{\nabla_{\mathscr{M}}\psi(z,\mathbf{x})}\, d\omega_{\mathscr{M}}\, dz \\ &+ \int_\Omega d(z)\, \varphi(z,\mathbf{x})\, \overline{\psi(z,\mathbf{x})}\, d\omega_{\mathscr{M}}\, dz\,,\end{aligned}$$

for $\varphi, \psi \in C_0^\infty(\Omega)$.

Thus, H and $\mathcal{E}$ are Hilbert spaces, equipped with their canonical norms: $\|\varphi\|_H^2 = \displaystyle\int_\Omega |\varphi(z,\mathbf{x})|^2 a(z)\, d\omega_{\mathscr{M}}\, dz$ and $\|\varphi\|_{\mathcal{E}}^2 = \|\varphi\|_H^2 + \mathfrak{b}(\varphi,\varphi)$. The operator A is well defined on $C_0^\infty(\Omega)$ and it is symmetric in H by definition.

We shall explore when A is a *q.e.s.a.* operator by using Theorem 2.2. Then the question is to determine under which conditions on the coefficients of A, $C_0^\infty(\Omega)$ is dense in $\mathcal{E}$. A related one is whether $C_c^\infty(\overline{\Omega}) \cap \mathcal{E}$ is dense in $\mathcal{E}$.

Notation 3.1. From now on, $\displaystyle\int_{z_0}$ and $\displaystyle\int^{z_1}$ respectively denote $\displaystyle\int_{z_0}^{z_0+\varepsilon}$ and $\displaystyle\int_{z_1-\varepsilon}^{z_1}$ for a positive and small enough ε.

Theorem 3.2. *Let A be the operator defined in (2). Then*

1. $C_c^\infty(\overline{\Omega}) \cap \mathcal{E}$ *is dense in* $\mathcal{E}$ *if and only if* $\displaystyle\int^\infty \left(\frac{1}{b(z)} + d(z) + a(z)\right) dz = +\infty$,
2. *A is a q.e.s.a. operator (i.e. $C_0^\infty(\Omega)$ is dense in $\mathcal{E}$) if and only if* $\displaystyle\int^\infty \left(\frac{1}{b(z)} + d(z) + a(z)\right) dz = +\infty$ *and* $\displaystyle\int_0 \left(\frac{1}{b(z)} + d(z) + a(z)\right) dz = +\infty$.

Proof:

The proof goes through three steps: first reducing the problem to a one dimensional case, second proving that compactly supported functions are dense under the given hypotheses, and finally getting the desired result.

Step 1: reduction to the one dimensional case. Let $\{\lambda_k,\, k \geq 0\}$ be the spectrum of $-\Delta_{\mathscr{M}}$, with $\lambda_0 = 0$ and λ_k an increasing sequence, and let $(\psi_k)_{k\geq 0}$ be an associated orthonormal basis of $L^2(\mathscr{M})$.

We define, for each $k \geq 0$,

$$A_k\, u(z) = \frac{1}{a(z)} \left(-\Big(b(z)\, u'(z)\Big)' + \Big(\lambda_k\, c(z) + d(z)\Big)\, u(z) \right), \tag{3}$$

for $u \in C_0^\infty\Big((0,+\infty)\Big)$, with the underlying Hilbert space $H_0 = L^2\Big((0,+\infty), a(z)\,dz\Big)$ and energy spaces $\mathcal{E}_k = \Big\{u \in H_0 \cap H^1_{loc}\Big((0,+\infty)\Big) : \mathfrak{b}_k(u,u) < +\infty\Big\}$, where

$$\mathfrak{b}_k(u,v) = \int_0^{+\infty} b(z)\, u'(z)\, \overline{v'(z)}\, dz + \int_0^{+\infty} \Big(\lambda_k\, c(z) + d(z)\Big)\, u(z)\, \overline{v(z)}\, dz\,.$$

Then we consider the Hilbert spaces $\mathcal{E}_k$ with their natural norms

$$\|u\|^2_{\mathcal{E}_k} = \int_0^{+\infty} b(z)\, |u'(z)|^2\, dz + \int_0^{+\infty} \Big(\lambda_k\, c(z) + d(z) + a(z)\Big)\, |u(z)|^2\, dz.$$

Lemma 3.3. *$C_c^\infty(\overline{\Omega}) \cap \mathcal{E}$ (respectively $C_0^\infty(\Omega)$) is dense in $\mathcal{E}$ if and only if $C_c^\infty\Big([0,\infty)\Big) \cap \mathcal{E}_k$ (respectively $C_0^\infty\Big((0,+\infty)\Big)$ is dense in $\mathcal{E}_k$ for all $k \geq 0$.*

Proof:

Given $\varphi \in \mathcal{E}$, it can be decomposed into a sum $\varphi = \sum_{k\geq 0} u_k \otimes \psi_k$, where $u_k \in \mathcal{E}_k$ and

$$\|\varphi\|^2_{\mathcal{E}} = \sum_{k\geq 0} \|u_k\|^2_{\mathcal{E}_k}.$$

So, density in $\mathcal{E}$ implies density in each $\mathcal{E}_k$.

For the reciprocal, given $\varphi \in \mathcal{E}$ we first approximate it by the functions $\varphi_m = \sum_{k=0}^{m} u_k \otimes \psi_k$, and density in $\mathcal{E}_k$ for all $k \geq 0$ implies that each φ_m can be approximate by functions of $C_c^\infty(\Omega) \cap \mathcal{E}$ (respectively $C_0^\infty(\Omega)$). □

Step 2: density of compactly supported functions in $\mathcal{E}_0$. Here, for convenience we shall restrict our attention at first to the case $k = 0$ and $d(z) \equiv 0$.

We define

$$\mathcal{E}_{0,c} = \mathcal{E}_0 \cap \{\text{functions with compact support in } [\,0,+\infty)\},$$
$$\mathcal{E}_{0,0} = \mathcal{E}_0 \cap \{\text{functions with compact support in } (0,+\infty)\}.$$

Lemma 3.4. *$\mathcal{E}_{0,c}$ is dense in $\mathcal{E}_0$ if and only if* $\displaystyle\int^{+\infty} \left(\frac{1}{b(z)} + a(z)\right) dz = +\infty$.

Proof:

Assume first that $\int^{+\infty}\left(\frac{1}{b(z)}+a(z)\right)dz<+\infty$. If $u\in\mathcal{E}_0$, then $u'\in L^1\Big([z',+\infty)\Big)$ for any $z'>0$, since $\int_{z'}^{+\infty}\frac{1}{b(z)}\,dz<+\infty$ and using Hölder inequality. Moreover $\lim_{z\to\infty}u(z)$ exists and is not necessarily zero because $\int^{+\infty}a(z)<+\infty$. Thus, there exists a linear functional on $\mathcal{E}_0$ which vanishes on $\mathcal{E}_{0,c}$ but not everywhere, showing that $\mathcal{E}_{0,c}$ is not dense in $\mathcal{E}_0$. Such functional may be

$$\lambda(u)=\int_0^{+\infty}\Big(u(z)\,\eta(z)\Big)'dz,$$

where $\eta(z)$ is a smooth function such that $\eta(z)=0$ if $z\in[0,z']$ and $\eta(z)=1$ if $z\geq 2z'$.

Assuming now that $\int^{+\infty}\left(\frac{1}{b(z)}+a(z)\right)dz=+\infty$, we shall see that $\mathcal{E}_{0,c}$ is dense in $\mathcal{E}_0$.

If there exists $z'>0$ such that $\int_{z'}^{+\infty}\frac{1}{b(z)}\,dz<+\infty$, taking $u\in\mathcal{E}_0$, we have again that $u'\in L^1\Big([z',+\infty)\Big)$, but now $\lim_{z\to+\infty}u(z)=0$ necessarily, since $\int^{+\infty}a(z)\,dz=+\infty$. Thus, we have

$$u(z)=-\int_z^{+\infty}u'(s)\,ds.$$

Hence, defining $\beta_0(z)=\int_z^{+\infty}\frac{1}{b(z)}\,dz$ and using Hölder inequality we have

$$|u(z)|\leq\sqrt{\beta_0(z)}\left(\int_z^{+\infty}b(z)\,|u'(z)|^2\,dz\right)^{1/2}. \tag{4}$$

Since $\|u\|_{\mathcal{E}_0}<+\infty$, for $\varepsilon>0$, there exists $z_0>0$ such that

$$\int_{z_0}^{+\infty}\Big(b(z)\,|u'(z)|^2+a(z)\,|u(z)|^2\Big)\,dz\leq\varepsilon. \tag{5}$$

Define $\chi(z)$ on $[0,+\infty)$ by

$$\chi(z)=\begin{cases}1 & \text{if}\quad 0\leq z\leq z_0\\ \ln\left(\frac{\beta_0(z)}{\beta_0(z_1)}\right) & \text{if}\quad z_0\leq z\leq z_1\\ 0 & \text{if}\quad z_1\leq z\leq+\infty\end{cases}$$

with z_1 given by the equation $\beta_0(z_1)=\mathrm{e}^{-1}\,\beta_0(\mathrm{z}_0)$. Then we have

$$\begin{aligned}\|u-u\,\chi\|_{\mathcal{E}_0}^2 &\leq \int_{z_0}^{+\infty}a(z)\Big(1-\chi(z)\Big)^2|u(z)|^2\,dz+\int_{z_0}^{+\infty}b(z)\Big(1-\chi(z)\Big)^2|u'(z)|^2\,dz\\ &+\int_{z_0}^{+\infty}b(z)\,\chi'(z)^2\,|u(z)|^2\,dz\,.\end{aligned}$$

The first two terms are small by (5), and for the third one, we have from (4) and (5)

$$\begin{aligned}\int_{z_0}^{+\infty}b(z)\,\chi'(z)^2\,u(z)^2\,dz &\leq \int_{z_0}^{z_1}\frac{1}{b(z)\;\beta_0(z)^2}\,|u(z)|^2\,dz\\ &\leq \varepsilon\int_{z_0}^{z_1}\frac{1}{b(z)\;\beta_0(z)}\,dz\\ &\leq C\varepsilon\,.\end{aligned}$$

Since $u\,\chi \in \mathcal{E}_{0,c}$, the density of $\mathcal{E}_{0,c}$ in $\mathcal{E}_0$ is proved.

For the case when $\displaystyle\int^{+\infty} \frac{1}{b(z)}\,dz = +\infty$, given $z' > 0$ we define $\beta_0(z) = \displaystyle\int_{z'}^{z} \frac{1}{b(s)}\,ds$, and we choose z^*, z such that $z' \le z^* \le z$. We have

$$|u(z) - u(z^*)| \le \int_{z^*}^{z} |u'(s)|\,ds \le \left(\int_{z^*}^{+\infty} b(s)\,|u'(s)|^2\,ds\right)^{\frac{1}{2}} \sqrt{\beta_0(z)},$$

hence

$$|u(z)| \le |u(z^*)| + \left(\int_{z^*}^{+\infty} b(s)\,|u'(s)|^2\,ds\right)^{\frac{1}{2}} \sqrt{\beta_0(z)}.$$

This implies

$$\lim_{z\to+\infty} \frac{|u(z)|}{\sqrt{\beta_0(z)}} = 0\,. \tag{6}$$

Now, by (6) for any $\varepsilon > 0$, there exists $z_0 > 0$ such that

$$\frac{|u(z_0)|^2}{\beta_0(z_0)} + \int_{z_0}^{+\infty} \Big(b(z)\,|u'(z)|^2 + a(z)\,|u(z)|^2\Big)\,dz \le \varepsilon.$$

Then,

$$|u(z)| \le |u(z_0)| + \sqrt{\varepsilon}\sqrt{\beta_0(z)},$$

when $z \ge z_0$. We define $\chi(z)$ by

$$\chi(z) = \begin{cases} 1 & \text{if} \quad 0 \le z \le z_0 \\ \ln\left(\frac{\beta_0(z_1)}{\beta_0(z)}\right) & \text{if} \quad z_0 \le z \le z_1 \\ 0 & \text{if} \quad z_1 \le z \le +\infty \end{cases}$$

with z_1 given by the equation $\beta(z_1) = \mathrm{e}\,\beta_0(\mathrm{z}_0)$, and we can prove, as above, that there exists a constant C such that

$$\|u - u\chi\|^2_{\mathcal{E}_0} \le C\,\varepsilon\,.$$

Thus, in this case also, $\mathcal{E}_{0,c}$ is dense in $\mathcal{E}_0$. ☐

Lemma 3.5. (1) *The set of all $u \in \mathcal{E}_0$ which vanishes in some neighbourhood of 0 (depending on u) is dense in $\mathcal{E}_0$ if and only if $\displaystyle\int_0 \left(\frac{1}{b(z)} + a(z)\right)\,dz = +\infty$*

(2) *$\mathcal{E}_{0,0}$ is dense in $\mathcal{E}$ if and only if $\displaystyle\int^{\infty} \left(\frac{1}{b(z)} + a(z)\right)\,dz = +\infty$ and $\displaystyle\int_0 \left(\frac{1}{b(z)} + a(z)\right)\,dz = +\infty.$*

Proof:

(i) We consider the transformation $\phi(z) = \dfrac{1}{z} : (0,+\infty) \to (0,+\infty)$, and let

$$\mathcal{E}_\phi = \Big\{u \in H^1_{loc}((0,+\infty) : \|u\|^2_\phi = \int_0^{+\infty} \Big(b_\phi(z)\,|u'(z)|^2 + a_\phi(z)\,|u(z)|^2\Big)\,dz < +\infty\Big\},$$

where $b_\phi(z) = z^2\,b\,(1/z)$ and $a_\phi(z) = a\,(1/z)\,/z^2$.

Then $\mathcal{E}_\phi$ and $\mathcal{E}_0$ are isomorphic, through the application $\Phi : \mathcal{E}_0 \to \mathcal{E}_\phi$ given by $\Phi(v) = u = v \circ \phi$.

By lemma 3.4, $\mathcal{E}_{\phi,c}$ is dense in $\mathcal{E}_\phi$ if and only if $\displaystyle\int^{+\infty}\left(\frac{1}{b_\phi(z)}+a_\phi(z)\right)dz=\int_0\left(\frac{1}{b(z)}+a(z)\right)dz=\infty$, and we observe that $v\in\mathcal{E}_0$ vanishes in a neighbourhood of 0 if and only if $\Phi(v)\in\mathcal{E}_{\phi,c}$.

(ii) follows directly from both assertion *(i)* and lemma 3.4. $\square$

Step 3: conclusion in the one dimensional case.

Lemma 3.6.

(1) $C_c^\infty\Big([0,+\infty)\Big)\cap\mathcal{E}_0$ *is dense in* $\mathcal{E}_0$ *if and only if* $\displaystyle\int^{+\infty}\left(\frac{1}{b(z)}+a(z)\right)dz=+\infty$*,*

(2) $C_0^\infty\Big((0,+\infty)\Big)$ *is dense in* $\mathcal{E}_0$ *if and only if* $\displaystyle\int^{+\infty}\left(\frac{1}{b(z)}+a(z)\right)dz=+\infty$ *and* $\displaystyle\int_0\Big(\frac{1}{b(z)}+a(z)\Big)\,dz=+\infty$.

Proof.

(ii) Assume first $C_0^\infty\Big((0,+\infty)\Big)$ is dense in $\mathcal{E}_0$, then $\mathcal{E}_{0,0}$ must be dense too, and this implies, by lemma 3.5, $\displaystyle\int^{+\infty}\left(\frac{1}{b(z)}+a(z)\right)dz=\int_0\left(\frac{1}{b(z)}+a(z)\right)dz=+\infty$.

Reciprocally, if $\displaystyle\int^{+\infty}\left(\frac{1}{b(z)}+a(z)\right)dz=\int_0\left(\frac{1}{b(z)}+a(z)\right)dz=+\infty$, by lemma 3.5, $\mathcal{E}_{0,0}$ is dense in $\mathcal{E}_0$. Therefore it suffices to prove that $C_0^\infty\Big((0,+\infty)\Big)$ is dense in $\mathcal{E}_{0,0}$. For this purpose we will show that for any compact interval $I=[z_0,z_1]\subset(0,+\infty)$, $C_0^\infty(I)$ is dense in $\mathcal{E}_I=\{u\in\mathcal{E}_0:\ \operatorname{supp}u\subset I\}$.

Let $m=\displaystyle\int_I b(z)\,dz$ and define $\phi:I\to J=[0,m]$ by $\phi(z)=\displaystyle\int_{z_0}^z b(s)\,ds$. Then, $L^2\Big(I,b(z)dz\Big)$ and $L^2(J,ds)$ are isomorphic through the application $\Phi:L^2(J,ds)\to L^2\Big(I,b(z)dz\Big)$ such that $\Phi(v)=v\circ\phi$.

Let $u\in\mathcal{E}_I$, and denote $f=u'$ and $g=f\circ\phi^{-1}$, $g\in L^2(J,ds)$, then there exists a sequence $(g_n)_{n\geq 0}$ such that $g_n\in C_0(\mathring{J})$ for all $n\geq 0$ and $g_n\to g$ in $L^2(J,ds)$. Let $f_n=g_n\circ\phi$, then $f_n\in C_0(\mathring{I})$ and $f_n\to f$ in $L^2\Big(I,b(z)dz\Big)$, we also have that

$$\int_I\Big|f(z)-f_n(z)\Big|\,dz\leq C\left(\int_I b(z)\,\Big|f(z)-f_n(z)\Big|^2\,dz\right)^{\frac12},$$

by Cauchy-Schwarz inequality and because $\dfrac{1}{b}\in L^1_{loc}\Big((0,\infty)\Big)$. Since $\displaystyle\int_I f(z)\,dz=0$, we deduce that

$$\lim_{n\to\infty}\int_I f_n(z)\,dz=0.$$

Choose $\chi\in C_0(\mathring{I})$, such that $\displaystyle\int_I\chi(z)\,dz=1$, and define

$$\tilde{f}_n=f_n-\left(\int_I f_n(z)dz\right)\chi.$$

Then $\displaystyle\int_I\tilde{f}_n(z)\,dz=0$, $\tilde{f}_n\in C_0(\mathring{I})$ and $\tilde{f}_n\to f$ in $L^2\Big(I,b(z)dz\Big)$:

$$\begin{aligned}\int_I b(z)\left(f(z)-\tilde{f}_n(z)\right)^2dz\quad&\leq\textstyle\int_I b(z)\left(f(z)-f_n(z)\right)^2dz+\left(\int_I f_n(z)\,dz\right)^2\int_I b(z)\,\chi(z)^2\,dz\\ &\underset{n\to\infty}{\longrightarrow}0\,.\end{aligned}\tag{7}$$

Set

$$\tilde{u}_n(z) = \int_{z_0}^{z} \tilde{f}_n(s)\, ds,$$

since $\int_I \tilde{f}_n(z)\, dz = 0$, $\tilde{u}_n(z) \in C_0^1(\mathring{I})$ for all $n \geq 0$, and by (7),

$$\lim_{n\to\infty} \int_I b(z) \left| u'(z) - u_n'(z) \right|^2 dz = 0,$$

and

$$\lim_{n\to\infty} \|u' - u_n'\|_\infty = 0$$

because

$$\lim_{n\to\infty} \int_I \left| f(z) - \tilde{f}_n(z) \right| \, dz = 0.$$

Hence we have

$$\lim_{n\to\infty} \int_I a(z) \left| u(z) - u_n(z) \right|^2 dz = 0,$$

so that, finally,

$$\lim_{n\to\infty} \|u - u_n\|_{\mathcal{E}_I} = 0.$$

This proves the density of $C_0^1(I)$ in $\mathcal{E}_I$. To pass from $C_0^1(I)$ to $C_0^\infty(I)$, a classical regularization procedure is enough: it shows that $C_0^\infty(I)$ is dense in $C_0^1(I)$ for the topology given by the norm

$$\sup_{z\in I} |u(z)| + \sup_{z \in I} |u'(z)|;$$

since a and b are integrable on I, this implies the same density for the topology induced by $\mathcal{E}_I$, and part *(ii)* of the lemma is completely proved.

Regarding part *(i)*, we will be sketchy. The necesity of the condition $\int^{+\infty} \left(\frac{1}{b(z)} + a(z) \right) \, dz = +\infty$ follows from lemma 3.4. Its sufficiency needs only to be proved when $C_0^\infty \Big((0,\infty) \Big)$ is not dense, that is to say when $\int_0 \left(\frac{1}{b(z)} + a(z) \right) \, dz < +\infty$.

But then, the same proof as above works, even when $I = [0, z_1]$. □

When $d(z) \neq 0$ all the above results straightforwardly follow by changing $a(z)$ in $a(z) + d(z)$.

Proof of theorem 3.2.

Let us now prove theorem 3.2 *(ii)*: if $C_0^\infty(\Omega)$ is dense in $\mathcal{E}$, by lemma 3.3 $C_0^\infty \Big((0,+\infty) \Big)$ is dense in $\mathcal{E}_k$ for all $k \geq 0$, in particular for $k = 0$, then by lemma 3.6, we have $\int^{+\infty} \left(\frac{1}{b(z)} + d(z) + a(z) \right) dz = \int_0 \left(\frac{1}{b(z)} + d(z) + a(z) \right) dz = +\infty$.

Conversely, if $\int^{+\infty} \left(\frac{1}{b(z)} + d(z) + a(z) \right) dz = \int_0 \left(\frac{1}{b(z)} + d(z) + a(z) \right) dz = +\infty$, we also have

$$\int^{+\infty} \left(\frac{1}{b(z)} + d(z) + a(z) + \lambda_k \, c(z) \right) dz = \int_0 \left(\frac{1}{b(z)} + d(z) + a(z) + \lambda_k \, c(z) \right) dz = +\infty \,,$$

then $C_0^\infty \Big((0,+\infty) \Big)$ is dense in $\mathcal{E}_k$ for all k, we can see it changing $a(z)$ by $d(z) + a(z) + \lambda_k \, c(z)$ in all the previous results, and again by lemma 3.3, $C_0^\infty(\Omega)$ is dense in $\mathcal{E}$.

The proof of *(i)* analogously follows. Theorem 3.2 is completely proved. □

Remark 3.7. Under different hypotheses, when the coefficients of the operator A depend on $(z, \mathbf{x})$ we have given a characterization of *q.e.s.a.* operators in [1]. Warning: in page 21 of that reference, the integrand of (43) was mistakenly written as $\frac{1}{M_{n+1,n+1}(z,\mathbf{x})}$ instead of $(M^{-1})_{n+1,n+1}(z, \mathbf{x})$.

Essentially selfadjointness characterization. The characterization of *e.s.a.* for the operator A defined in (2) will rely on the real-valued solutions of the O.D.E.

$$-\Big(b(z)\, u'(z)\Big)' + d(z)\, u(z) = 0 \tag{8}$$

on $(0, z')$ and on $(z', +\infty)$.

A typical case is when $\int_0 a(z)\, dz < +\infty$, but $\int^{+\infty} a(z)\, dz = +\infty$. Then since we may assume A to be *q.e.s.a.* (otherwise it cannot be *e.s.a.*), we have $\int_0 \left(\frac{1}{b(z)} + d(z)\right) dz = +\infty$. In such a case, we will show that there is a unique solution of (8), denoted by α, such that

$$\begin{cases} \alpha \text{ is a solution of (8) in } (0, z'), \\ \alpha(z') = 1, \\ \int_0^{z'} \Big(b(z)\, \alpha'(z)^2 + d(z)\, \alpha(z)^2\Big)\, dz < +\infty\ . \end{cases} \tag{9}$$

Then, we define $\beta(z)$, $z \in (0, z')$, by

$$\beta(z) = \alpha(z) \int_z^{z'} \frac{1}{b(s)\, \alpha(s)^2}\, ds\, . \tag{10}$$

Note that, by construction, β is another solution of (8) in $(0, z')$. We shall prove that: A is *e.s.a.* if and only if $\int_0 \beta(z)^2\, a(z)\, dz = +\infty$.

In the case where the role of 0 and $+\infty$ are exchanged, the result is similar. We will show that there exists a unique function α such that

$$\begin{cases} \alpha(z) \text{ is a solution of (8) in } (z', +\infty), \\ \alpha(z') = 1, \\ \int_{z'}^{+\infty} \Big(b(z)\, \alpha'(z)^2 + d(z)\, \alpha(z)^2\Big)\, dz < +\infty\ . \end{cases} \tag{11}$$

Then, we define $\beta(z)$, $z \in (z', +\infty)$, by

$$\beta(z) = \alpha(z) \int_{z'}^{z} \frac{1}{b(s)\, \alpha(s)^2}\, ds\ , \tag{12}$$

and we shall prove that: A is *e.s.a.* if and only if $\int^{+\infty} \beta(z)^2\, a(z)\, dz = +\infty$.

Note that, when $d(z) \equiv 0$ the problem considerably simplifies since, in this case, $\alpha \equiv 1$ and $\beta(z)$ turns out to be either $\beta_0(z) = \int_z^{z'} \frac{1}{b(z)}\, dz$ or $\beta_0(z) = \int_{z'}^{z} \frac{1}{b(z)} dz$ respectively.

Notation 3.8. We denote $\Big(\alpha(z), \beta(z)\Big)$ the above couples of solutions of (8); the context will indicate whether $z \in (0, z')$, in which case $\Big(\alpha(z), \beta(z)\Big)$ are given by (9) and (10), or $z \in (z', +\infty)$, where $\Big(\alpha(z), \beta(z)\Big)$ are given by (11) and (12).

With this notation, the result is the following.

Theorem 3.9. *Assume the operator A given in (2) to be q.e.s.a., that is to say*

$$\int_0 \left(\frac{1}{b(z)} + d(z) + a(z)\right) dz = \int^{+\infty} \left(\frac{1}{b(z)} + d(z) + a(z)\right) dz = +\infty \ .$$

There are four cases:

(1) *If* $\int_0 a(z)\, dz = \int^{+\infty} a(z)\, dz = +\infty$*, then A is e.s.a.;*

(2) *If* $\int_0 a(z)\, dz < +\infty$ *and* $\int^{+\infty} a(z)\, dz = +\infty$*, then A is e.s.a. if and only if* $\int_0 \beta(z)^2 a(z)\, dz = +\infty$ *;*

(3) *If* $\int_0 a(z)\, dz = +\infty$ *and* $\int^{+\infty} a(z)\, dz < +\infty$*, then A is e.s.a. if and only if* $\int^{+\infty} \beta(z)^2 a(z)\, dz = +\infty$ *;*

(4) *If* $\int_0 a(z)\, dz < +\infty$ *and* $\int^{+\infty} a(z)\, dz < +\infty$*, then A is e.s.a. if and only if* $\int_0 \beta(z)^2 a(z)\, dz = \int^{+\infty} \beta(z)^2 a(z)\, dz = +\infty$ *.*

Remark 3.10. Take care of the uniqueness of α (and thus the meaningfulness of the definitions above): it holds when $\int_0 \left(\frac{1}{b(z)} + d(z)\right) dz = +\infty$ or $\int^{+\infty} \left(\frac{1}{b(z)} + d(z)\right) dz = +\infty$, according to where the variable z lives.

Preliminary step: study of solutions of (8).

Lemma 3.11. *Let $u(z)$ be a solution of (8) in some interval $I \subset (0, +\infty)$. Then the function $b(z)\, u(z)'\, u(z)$ is increasing in I.*

Proof:

From (8) we obtain

$$-\Big(b(z)\, u(z)'\, u(z)\Big)' + b(z)\, u'(z)^2 + d(z)\, u(z)^2 = 0\,,$$

showing that $\Big(b(z)\, u(z)'\, u(z)\Big)'$ is nonnegative. $\square$

Lemma 3.12. *Let $u(z)$ be a solution of (8) in $(0, z')$. Then*

$$\int_0^{z'} \Big(b(z)\, u'(z)^2 + d(z)\, u(z)^2\Big)\, dz = +\infty$$

if and only if

$$\lim_{z \to 0^+} b(z)\, u'(z)\, u(z) = -\infty\,.$$

Proof:

Since $u(z')$ and $u'(z')$ exist, the proof follows immediately from the fact that, for $0 < z_0 < z'$, we have

$$\int_{z_0}^{z'} \Big(b(z)\, u'(z)^2 + d(z)\, u(z)^2\Big)\, dz = \int_{z_0}^{z'} \Big(b(z)\, u(z)'\, u(z)\Big)'\, dz = b(z')\, u'(z')\, u(z') - b(z_0)\, u'(z_0)\, u(z_0)\,.$$

$\square$

Lemma 3.13. *Let $z'>0$ be chosen.*

(1) *There exists at least one solution $\alpha(z)$ of (8), in the interval $(0,z')$, such that*
$$\alpha(z')=1$$
and
$$\int_0^{z'}\Big(b(z)\,\alpha'(z)^2+d(z)\,\alpha(z)^2\Big)\,dz<+\infty\,.$$
This solution is positive and increasing in $(0,z')$, satisfying
$$\lim_{z\to 0^+} b(z)\,\alpha'(z)\,\alpha(z)=0\,.$$

(2) *If in addition $\displaystyle\int_0^{z'}\left(\frac{1}{b(z)}+d(z)\right)\,dz=+\infty$, this solution is unique.*

Proof:

Let $L^2_b\Big((0,z')\Big)$ be the space of measurable functions $f(z)$ such that
$$\int_0^{z'} b(z)\,f(z)^2\,dz<+\infty\,.$$
We define, for any f in this space, the function Tf by
$$Tf(z)=1-\int_z^{z'} f(s)\,ds\,,$$
so that $Tf\in C((0,z'))\bigcap H^1_{loc}\Big((0,z')\Big)$, with $(Tf)'(z)=f(z)$. Let
$$q(f)=\int_0^{z'}\left(b(z)\,f(z)^2+d(z)\,\Big(Tf(z)\Big)^2\right)\,dz,$$
taking values in $(0,+\infty]$, and
$$q_0=\inf_{f\in L^2_b} q(f)\,.$$
Note that q_0 is finite since, for example, for $f(z)=\dfrac{1}{z'-z_0}\mathbf{1}_{[z_0,z']}(z)$ for some $0<z_0<z'$, $q(f)<+\infty$. We shall show that q_0 is in fact a minimum. To this end, let $(f_n)_{n\in\mathbb{N}}$ be a minimising sequence
$$\lim_{n\to+\infty} q(f_n)=q_0\,.$$
Then, by construction,
$$\sup_{n\in\mathbb{N}}\|f_n\|_{L^2_b}<+\infty\,,$$
so that (up to extracting a subsequence) we may suppose that the sequence (f_n) has a weak limit f_0 in $L^2_b\Big((0,z')\Big)$. Let us prove that $q(f_0)=q_0$.

For any $z_0\in(0,z')$ and for all $z\geq z_0$
$$|Tf_n(z)|\leq 1+\left(\int_{z_0}^{z'}\frac{1}{b(z)}\,dz\right)^{1/2}\|f_n\|_{L^2_b}\leq C(z_0)\,,$$
and
$$Tf_0(z)\,\mathbf{1}_{[z_0,z']}(z)=\lim_{n\to+\infty} Tf_n(z)\,\mathbf{1}_{[z_0,z']}(z)\,.$$

So, by dominated convergence, we have

$$\lim_{n\to+\infty}\int_{z_0}^{z'} d(z)\left(Tf_n(z)\right)^2 dz=\int_{z_0}^{z'} d(z)\left(Tf_0(z)\right)^2 dz\,.$$

Also we know that

$$\int_{z_0}^{z'} b(z)\,f_0(z)^2\,dz\le\liminf_{n\to+\infty}\int_{z_0}^{z'} b(z)\,f_n(z)^2\,dz\,,$$

since $f_0=\underset{n\to+\infty}{\text{w-lim}}\,f_n$ in $L_b^2\left((0,z')\right)$ as well. From these two facts, we deduce

$$\int_{z_0}^{z'}\left(b(z)\,f_0(z)^2+d(z)\left(Tf_0(z)\right)^2\right)dz$$

$$\le\liminf_{n\to+\infty}\int_{z_0}^{z'}\left(b(z)\,f_n(z)^2+d(z)\left(Tf_n(z)\right)^2\right)dz\le q_0\,.$$

Letting $z_0\to 0^+$, we obtain $q(f_0)\le q_0$, and thus $q(f_0)=q_0$ as desired.

Let now $\alpha(z)=Tf_0(z)$. For any $u\in C\left((0,z')\right)\bigcap H^1_{loc}\left((0,z')\right)$, with $u(z')=1$, define

$$\begin{aligned}Q(u)&=q(u')\\&=\textstyle\int_0^{z'}\left(b(z)\,u'(z)^2+d(z)\,u(z)^2\right)dz\,.\end{aligned}$$

We have proved that

$$Q(\alpha)=\min_u Q(u)\,.$$

But since $Q(\alpha^+)\le Q(\alpha)$ with strict inequality if and only if $\alpha^-\neq 0$, and since $\alpha^+\in C\left((0,z')\right)\bigcap H^1_{loc}\left((0,z')\right)$ with $\alpha^+(z')=1$, we must have

$$Q(\alpha^+)=Q(\alpha)\,,$$

and $\alpha^+=\alpha$, i.e., α is positive in $(0,z']$.

If $\psi\in C\left((0,z')\right)\bigcap H^1_{loc}\left((0,z')\right)$ is such that $Q(\alpha+t\psi)<+\infty$ for all $t\in\mathbb{R}$ and $\psi(z')=0$, we must have

$$Q(\alpha)\le Q(\alpha+t\psi)\,,$$

and this implies

$$\int_0^{z'}\left(b(z)\,\alpha(z)'\psi(z)'+d(z)\,\alpha(z)\,\psi(z)\right)dz=0\,.$$

This, in particular, is true for all $\psi\in C_0^\infty\left((0,z')\right)$, implying that

$$-\left(b(z)\,\alpha(z)'\right)'+d(z)\,\alpha(z)=0$$

in $(0,z')$.

But then, this means that

$$\int_0^{z'}\left(b(z)\,\alpha(z)'\psi(z)'+\left(b(z)\,\alpha(z)'\right)'\psi(z)\right)dz=0$$

for all $\psi\in C\left((0,z')\right)\cap H^1_{loc}\left((0,z')\right)$ with $Q(\alpha+t\psi)<+\infty$ and $\psi(z')=0$. Therefore

$$\lim_{z\to 0^+} b(z)\,\alpha'(z)\,\psi(z)=0\,.$$

Choosing $\psi = \alpha\,\eta$, where $\eta \in C^\infty\Big(0,+\infty)\Big)$, $\eta = 1$ near 0 and $\eta = 0$ near z', we get

$$\lim_{z\to 0^+} b(z)\,\alpha'(z)\,\alpha(z) = 0\,.$$

With lemma 3.11, this shows that (recall that α is positive) α^2 and hence α are both increasing in $(0,1)$. Thus, part *(i)* is entirely proved.

(ii) Let

$$\beta(z) = \alpha(z)\int_z^{z'} \frac{1}{b(s)\,\alpha(s)^2}\;ds\ .$$

Then, $\beta(z)$ is another solution of (8) in $(0,z')$, so that any solution writes $\lambda\,\alpha(z) + \mu\,\beta(z)$, $\lambda,\mu \in \mathbb{R}$. The uniqueness of $\alpha(z)$ will follow from the proof of

$$\int_0^{z'} \Big(b(z)\,\beta'(z)^2 + d(z)\,\beta(z)^2\Big)\,dz = +\infty\,.$$

A direct calculation shows that $\beta(z') = 0$ and $\beta'(z') = -\dfrac{1}{b(z')}$. Thus, from the O.D.E. (8), we obtain

$$-\beta'(z) = \frac{1}{b(z)} + \frac{1}{b(z)}\int_z^{z'} d(s)\,\beta(s)\;ds\ .$$

Since β is positive by construction, it turns out to be decreasing in $(0,z')$, with

$$|\beta'(z)| \geq \frac{1}{b(z)}\ , \qquad 0 < z \leq z'\,,$$

and

$$\beta(z) \geq \int_z^{z'} \frac{1}{b(s)}\,ds\ =: \beta_0(z)\,.$$

Hence, there exists a constant C such that $\beta(z) \geq C$ if $z \leq z'/2$, and we obtain

$$\begin{aligned}\int_0^{z'} \Big(b(z)\,\beta'(z)^2 + d(z)\,\beta(z)^2\Big)\,dz \quad &\geq \textstyle\int_0^{z'} \frac{1}{b(z)}\,dz + C^2\int_0^{z'/2} d(z)\,dz \\ &= +\infty\,.\end{aligned}$$

The lemma is proved. $\square$

Lemma 3.13 has an analogous counterpart near $+\infty$, which is the following.

Lemma 3.14. *Let $z' > 0$ be chosen.*

(1) *There exists at least one solution $\alpha(z)$ of (8), in the interval $(z',+\infty)$, such that*

$$\alpha(z') = 1$$

and

$$\int_{z'}^{+\infty} \Big(b(z)\,\alpha'(z)^2 + d(z)\,\alpha(z)^2\Big)\,dz < +\infty\,.$$

This solution is positive and increasing in $(z',+\infty)$, satisfying

$$\lim_{z\to +\infty} b(z)\,\alpha'(z)\,\alpha(z) = 0\,.$$

(2) *If in addition $\displaystyle\int^{+\infty}\left(\frac{1}{b(z)} + d(z)\right)\,dz = +\infty$, this solution is unique.*

Proof:

By making the change of variable $z \mapsto \dfrac{z'^2}{z}$, the proof immediately follows from the previous lemma. $\square$

Remark 3.15. The function $\alpha(z)$ given in $(0, z')$ (respectively in $(z', +\infty)$) by lemma 3.13 (resp. lemma 3.14) is not a solution of (8) on $(0, +\infty)$, but of

$$-\Big(b(z)\,\alpha'(z)\Big)' + d(z)\,\alpha(z) = \lambda\,\delta_{z'}(z)\,,$$

where $\delta_{z'}(z)$ is the Dirac measure at $z = z'$, and $\lambda = \displaystyle\int_0^{+\infty} \Big(b(z)\,\alpha'(z)^2 + d(z)\,\alpha(z)^2\Big)\,dz$.

Main step: *e.s.a.* **characterization in dimension one.** Let us consider now the operator

$$A_0\,u(z) = \frac{1}{a(z)}\left(-\Big(b(z)\,u'(z)\Big)' + d(z)u(z)\right)$$

defined as in (3) with

$$\int_0 \left(\frac{1}{b(z)} + d(z)\right)\,dz = \int^{+\infty} \left(\frac{1}{b(z)} + d(z)\right)\,dz = +\infty\,.$$

Lemma 3.16. *If* $\displaystyle\int_0 a(z)\,dz = \int^{+\infty} a(z)\,dz = +\infty$, *$A_0$ is an e.s.a. operator.*

Proof:

Assume A_0 is not *e.s.a.*. By lemma 2.5 there exists $u \in H_0$ such that

$$-\Big(b(z)\,u'(z)\Big)' + d(z)\,u(z) + a(z)\,u(z) = 0\,,$$

and $u \notin \mathcal{E}_0$, i.e., either $\displaystyle\int_0 \Big(b(z)(u'(z))^2 + \Big(d(z) + a(z)\Big)\,u(z)^2\Big)\,dz = +\infty$ or $\displaystyle\int^{+\infty} \Big(b(z)(u'(z))^2 + \Big(d(z) + a(z)\Big)\,u(z)^2\Big)\,dz = +\infty$ (or both).

If $\displaystyle\int_0 \left(b(z)\,u'(z)^2 + \Big(d(z) + a(z)\Big)\,u(z)^2\right)\,dz = +\infty$, by lemma 3.12 (changing d in $d + a$) we have

$$\lim_{z \to 0^+} b(z)\,u'(z)\,u(z) = -\infty\,.$$

In particular, $u'(z)\,u(z) < 0$ for $z \le z_0$, for some $z_0 > 0$, so that u^2 is decreasing in $(0, z_0]$. But since $\displaystyle\int_0^{+\infty} a(z)\,u(z)^2\,dz < +\infty$, this implies $\displaystyle\int_0 a(z)\,dz < +\infty$, which is a contradiction.

If $\displaystyle\int^{+\infty} \left(b(z)\,u'(z)^2 + \Big(d(z) + a(z)\Big)\,u(z)^2\right)\,dz = +\infty$, a change of variable reduces the proof to the preceding case. $\square$

Lemma 3.17. *Assume* $\displaystyle\int_0 a(z)\,dz < +\infty$ *and* $\displaystyle\int^{+\infty} a(z)\,dz = +\infty$. *Then, A_0 is an e.s.a. operator if and only if* $\displaystyle\int_0 \beta(z)^2 a(z)\,dz = +\infty$.

Proof:

We first assume that $\int_0 \beta(z)^2 a(z)\, dz < +\infty$. We set $u(z) = \beta(z)\,\eta(z)$ with $\eta \in C^\infty\Big([0,+\infty)\Big)$, $\eta = 1$ near 0 and $\eta = 0$ for $z \geq \varepsilon$. Then $u \in H_0$, $A_0^* u \in H_0$ but $u \notin \mathcal{E}_0$ by lemma 3.13. Thus A_0 is not *e.s.a.*.

Reciprocally, assume that A_0 is not *e.s.a.*. Then there exists $u \in H_0$ such that

$$-\Big(b(z)\, u'(z)\Big)' + d(z)\, u(z) + a(z)\, u(z) = 0\,,$$

and $u \notin \mathcal{E}_0$.

Since $\int^{+\infty} a(z)\, dz = +\infty$ and $\int^{+\infty} u(z)^2 a(z)\, dz < +\infty$, the same argument as in lemma 3.16 shows that necessarily

$$\int^{+\infty} \Big(b(z)\, u'(z)^2 + d(z)\, u(z)^2\Big)\, dz < +\infty\,.$$

Thus we must have

$$\int_0 \Big(b(z)\, u'(z)^2 + d(z)\, u(z)^2\Big)\, dz = +\infty\,.$$

By lemma 3.12, $\lim_{z\to 0^+} b(z)\, u'(z)\, u(z) = -\infty$, and in particular, u^2 is decreasing in $(0, z_0)$ for some $z_0 > 0$. We may assume that $u(z_0) > 0$ and $u'(z_0) < 0$ (up to changing u in $-u$). Let C_1 and C_2 be two constants such that

$$\begin{cases} C_1\, \alpha(z_0) + C_2\, \beta(z_0) = u(z_0), \\ C_1\, \alpha'(z_0) + C_2\, \beta'(z_0) = u'(z_0)\,. \end{cases}$$

They exist because we know that the Wronskian $b(z)\Big(\alpha(z)\beta'(z) - \alpha'(z)\beta(z)\Big)$ is never vanishing [1]. Moreover, we must have $C_2 \neq 0$, otherwise $u(z_0)$ and $u'(z_0)$ would have the same sign (recall that α is positive and increasing, by lemma 3.13). We even have $C_2 > 0$[2].

Let $v(z) = u(z) - C_1\, \alpha(z) - C_2\, \beta(z)$. We have

$$-\Big(b(z)\, v'(z)\Big)' + d(z)\, v(z) + a(z)\, u(z) = 0\,,$$

with $v(z_0) = v'(z_0) = 0$, $u > 0$ in $(0, z_0]$. By classical arguments, v must be positive and decreasing in $(0, z_0]$:

- It is so in some neighborhood of z_0, because $\Big(b(z)\, v'(z)\Big)' > 0$ near z_0 and $v'(z_0) = 0$, so that $v'(z) < 0$ in $(z_0 - \epsilon, z_0)$;
- it cannot change its sense of variation in $(0, z_0)$ ($v(z_1) > 0$, $v'(z_1) = 0$, $v''(z_1) \leq 0$ at some $z_1 < z_0$ is impossible).

Hence, since $C_2 > 0$, we have

$$\beta(z) \leq \frac{1}{C_2}\,(u(z) - C_1\, \alpha(z))$$

in $(0, z_0]$. Since α is bounded, $\int_0 a(z)\, dz < +\infty$ and $u \in H$, this implies

$$\int_0 \beta(z)^2 a(z)\, dz < +\infty\,,$$

and the proof is finished. $\square$

[1] In fact, it is a constant, equal to -1.

[2] $C_2 = b(z_0)[u(z_0)\alpha'(z_0) - u'(z_0)\alpha(z_0)] > 0$

Lemma 3.18. *Assume* $\int_0 a(z)\,dz = +\infty$ *and* $\int^{+\infty} a(z)\,dz < +\infty$*. Then,* A_0 *is an e.s.a. operator if and only if* $\int^{+\infty} \beta(z)^2\, a(z)\,dz = +\infty$*.*

Proof:
The result follows by a change of variable and the preceding lemma. □

Lemma 3.19. *Assume* $\int_0 a(z)\,dz < +\infty$ *and* $\int^{+\infty} a(z)\,dz < +\infty$*. Then,* A_0 *is an e.s.a. operator if and only if* $\int^{+\infty} \beta(z)^2 a(z)\,dz = +\infty$*.*

Proof:
If A_0 is not *e.s.a.*, there exists $u \in H_0$ solution of

$$-\Big(b(z)\,u'(z)\Big)' + d(z)\,u(z) + a(z)\,u(z) = 0\,,$$

and either $\int_0 \Big(b(z)\,u'(z)^2 + d(z)\,u(z)^2\Big)\,dz = +\infty$ or $\int^{+\infty} \Big(b(z)\,u'(z)^2 + d(z)\,u(z)^2\Big)\,dz = +\infty$. Use the arguments of lemma 3.17 or lemma 3.18, depending on the case.

Reciprocally, as we have done in lemma 3.17, we consider $u(z) = \beta(z)\,\eta(z)$ for a suitable η and the result follows. □

Final step: reduction to the one-dimensional case. Defining the operators A_k as in (3), i.e.,

$$A_k\,u(z) = \frac{1}{a(z)}\left(-\Big(b(z)u'(z)\Big)' + \Big(\lambda_k\,c(z) + d(z)\Big)u(z)\right),$$

we have the following result:

Lemma 3.20. *A is an e.s.a. operator if and only if for all $k \geq 0$ A_k is an e.s.a. operator.*

Proof:
We use the notation introduced in step 1 of the proof of theorem 3.2. By Lemma 2.5, if A_k is not *e.s.a.*, there exists $u \in H_0$, $u \in D(A_k^*)$ but $u \notin \mathcal{E}_k$. This implies that $\varphi = u \otimes \psi_k \in D(A^*)$ and $\varphi \notin \mathcal{E}$, so that A is not *e.s.a.* .

Reciprocally, if A is not *e.s.a.*, there exists $\varphi \in H$ non vanishing, such that

$$A^*\varphi + \varphi = 0.$$

Decompose

$$\varphi = \sum_{k\geq 0} u_k \otimes \psi_k,$$

there exists k such that $u_k \neq 0$. If $\phi \in C_0^\infty\Big((0,\infty)\Big)$, we have

$$0 =< \varphi, A(\phi\otimes\psi_k) + \phi\otimes\psi_k >_H =< u_k, A_k\phi + \phi >_{H_0},$$

which means that $A_k^* u_k + u_k = 0$. Thus A_k is not *e.s.a.* by lemma 2.5 again. □

Proof of theorem 3.9.

(i) If $\int_0 a(z)\,dz = +\infty$ and $\int^{+\infty} a(z)\,dz = +\infty$, then $\int_0 \Big(a(z) + \lambda_k\, c(z)\Big)\,dz = +\infty$ and $\int^{+\infty} \Big(a(z) + \lambda_k\, c(z)\Big)\,dz = +\infty$, for all $k \geq 0$. Therefore A_k is *e.s.a.* by lemma 3.16 with a changed in $a + \lambda_k\, c(z)$, and by lemma 3.20 A is *e.s.a.*.

In the cases *(ii)*, *(iii)* and *(iv)* if A is *e.s.a.* it follows by lemma 3.20 that in particular A_0 is *e.s.a.*. Then lemmas 3.17, 3.18 and 3.19 give the result.

For the converse, let us take the case *(ii)*. If A is not *e.s.a.*, by lemma 3.20 there exists $k \geq 0$ such that A_k is not *e.s.a.*. Then by lemma 3.17

$$\int_0 \beta_k(z)^2 a(z)\,dz < +\infty, \tag{13}$$

where β_k is the solution of

$$-\Big(b(z)\,u'(z)\Big)' + (c(z)\lambda_k + d(z))\,u(z) = 0$$

on $(0, z')$ with Cauchy data $u(z') = 0$ and $u(z') = -\dfrac{1}{b(z')}$. A classical comparison principle, applied to the functions β_k and β, defined in (10), give us $0 \leq \beta \leq \beta_k$ on $(0, z')$. Then (13) implies

$$\int_0 \beta(z)^2 a(z)\,dz < +\infty\,,$$

as desired.

The other cases are analogous.

Theorem 3.9 is completely proved. $\square$

Remark 3.21. The precise definition of the function $\beta(z)$ is needed only for the sufficiency of the condition

$$\int_0 \beta(z)^2 a(z)\,dz < +\infty$$

for A to be *e.s.a.*. This is not used in the reciprocal, where the "massless-β"

$$\beta_0(z) = \int_z^{z'} \frac{1}{b(s)}\,ds$$

would have worked as well[3]. But, for the sufficiency, if we choose $u(z) = \beta_0(z)\,\eta(z)$ in lemma 3.17, with $\eta \in C^\infty([0, +\infty))$, $\eta = 1$ near 0 and $\eta = 0$ for $z \geq \dfrac{z'}{2}$, then

$$A_0^*\,u(z) = \frac{1}{a(z)}\left(-\Big(b(z)\,\beta_0(z)\,\eta'(z)\Big)' + d(z)\,\beta_0(z)\,\eta(z)\right),$$

and this belongs to H_0 only when

$$\int_0 d(z)^2\,\beta_0(z)^2 \frac{1}{a(z)}\,dz < +\infty\,.$$

This gives a necessary and sufficient condition for *e.s.a.* in terms of $\beta_0(z)$ only, not $\beta(z)$, when $\dfrac{d(z)}{a(z)}$ is bounded:

[3]Recall that we have showed the inequality $\beta(z) \geq \beta_0(z)$ in the course of proving lemma 3.13.

Corollary 3.22. *When* $\dfrac{d(z)}{a(z)}$ *is bounded near* 0, $\displaystyle\int_0 a(z)\,dz < +\infty$ *and* $\displaystyle\int^{+\infty} a(z)\,dz = +\infty$, *A is e.s.a. if and only if* $\displaystyle\int_0 \beta_0(z)^2 a(z)\,dz = +\infty$.

There are similar statements in the other cases.

Remark 3.23. The previous results in the domain $(z_0, z_1) \times \mathscr{M}$
In some relevant examples one is lead to consider $\Omega = (z_0, z_1) \times \mathscr{M}$, $0 \le z_0 \le z_1 \le \infty$, and a differential operator A defined as in (2) by

$$A\varphi(z,\mathbf{x}) = \frac{1}{a(z)}\left\{-\partial_z\Big(b(z)\partial_z\varphi(z,\mathbf{x})\Big) - c(z)\Delta_{\mathscr{M}}\varphi(z,\mathbf{x}) + d(z)\varphi(z,\mathbf{x})\right\},$$

for all $\varphi \in C_0^\infty(\Omega)$, where the functions a, b, and c satisfy the following hypotheses:

- a, c, $d \in L^1_{loc}\Big((z_0, z_1)\Big)$, $b \in C\Big((z_0, z_1)\Big)$
- $a > 0$, $b > 0$, $c > 0$ and $d \ge 0$ in (z_0, z_1)
- a^{-1}, b^{-1}, $c^{-1} \in L^1_{loc}\Big((z_0, z_1)\Big)$.

The previous results straightforwardly generalize to such a case. For the convenience of the reader, we state the two main theorems.

Theorem 3.24. *A is a q.e.s.a. operator in H if and only if* $\displaystyle\int^{z_1}\left(\frac{1}{b(z)} + d(z) + a(z)\right)dz = +\infty$ *and* $\displaystyle\int_{z_0}\left(\frac{1}{b(z)} + d(z) + a(z)\right)dz = +\infty$.

Theorem 3.25. *We assume A is a q.e.s.a. operator, There are four cases:*

1. *If* $\displaystyle\int_{z_0} a(z)\,dz = \int^{z_1} a(z)\,dz = +\infty$, *then A is e.s.a.;*
2. *If* $\displaystyle\int_{z_0} a(z)\,dz < +\infty$ *and* $\displaystyle\int^{z_1} a(z)\,dz = +\infty$, *then A is e.s.a. if and only if* $\displaystyle\int_{z_0} \beta(z)^2 a(z)\,dz = +\infty$ *;*
3. *If* $\displaystyle\int_{z_0} a(z)\,dz = +\infty$ *and* $\displaystyle\int^{z_1} a(z)\,dz < +\infty$, *then A is e.s.a. if and only if* $\displaystyle\int^{z_1} \beta(z)^2 a(z)\,dz = +\infty$ *;*
4. *If* $\displaystyle\int_{z_0} a(z)\,dz < +\infty$ *and* $\displaystyle\int^{z_1} a(z)\,dz < +\infty$, *then A is e.s.a. if and only if* $\displaystyle\int_{z_0} \beta(z)^2 a(z)\,dz = \int^{z_1} \beta(z)^2 a(z)\,dz = +\infty$.

A typical situation where these results apply is when $\displaystyle\int^{z_1}\!\left(\frac{1}{b(z)} + d(z) + a(z)\right)dz = +\infty$ but $\displaystyle\int_{z_0}\!\left(\frac{1}{b(z)} + d(z) + a(z)\right)dz < +\infty$. Then $C_0^\infty(\Omega)$ is not dense in $\mathcal{E}$, but the only non trivial linear forms continuous on $\mathcal{E}$, vanishing on $C_0^\infty(\Omega)$, are supported on $\{z_0\} \times \mathscr{M}$. This means that a boundary condition must be chosen at $z = z_0$, but not at $z = z_1$.

Moreover if we have, for example, $\displaystyle\int^{z_1} a(z)\,dz = +\infty$, the selfadjoint extension $\tilde{A}$, defined from A with an appropriate boundary condition at $z = z_0$, will be unique. In particular, considering null Dirichlet

boundary condition, $\tilde{A}$ will be the selfadjoint extension of A constructed from the restriction of the bilinear form to $\mathcal{E}^0$.

4. Well-posedness of the Cauchy problem

Let A and Ω be as in the previous section. We assume A to be at least q.e.s.a. but not necessarily e.s.a.; we denote in the same way its unique selfadjoint extension with finite energy. We take functions $f \in \mathcal{E}$ and $g \in H$ and consider the Cauchy problem

$$(P)\begin{cases} \partial_{tt}\varphi + A\varphi &= 0, \\ \varphi(0,\cdot) &= f, \\ \partial_t\varphi(0,\cdot) &= g. \end{cases}$$

Theorem 4.1. *Under the hypotheses above, the problem (P) has a unique solution*

$$\phi \in C([0,\infty);\mathcal{E}) \cap C^1([0,\infty);H),$$

and there exists a constant $C > 0$ such that

$$\forall\ t > 0 \quad \|\phi(t,\cdot)\|_{\mathcal{E}} + \|\partial_t\phi(t,\cdot)\|_H \leq C(\|f\|_{\mathcal{E}} + \|g\|_H).$$

In this case, the energy

$$E(\phi,t) = \frac{1}{2}\int_\Omega \left(a(z)\,(\partial_t\phi)^2 + b(z)\,(\partial_z\phi)^2 + c(z)\,|\nabla\phi|^2 + d(z)|\phi|^2\right) d\mu$$

is well-defined and conserved:

$$\forall\ t > 0 \quad E(\phi,t) = \frac{1}{2}\left(\|g\|_H^2 + b(f,f)\right).$$

Proof:

Given $f \in D$ and $g \in \mathcal{E}$, the solution of (P) is given by (see, for example, [6])

$$\phi(t,\cdot) = \cos\left(tA^{\frac{1}{2}}\right)f + A^{-\frac{1}{2}}\sin\left(tA^{\frac{1}{2}}\right)g. \tag{14}$$

Taking into account that $D(A^{\frac{1}{2}}) = \mathcal{E}$, we have $\phi(t,\cdot) \in D$ and $\partial_t\phi(t,\cdot) \in \mathcal{E}$. That $\phi(t,\cdot)$ and $\partial_t\phi(t,\cdot)$ are continuous vector-valued functions (in D and in $\mathcal{E}$ respectively) rely on a classical density argument we only sketch. For $\varepsilon > 0$ we set $f_\varepsilon = (I+\varepsilon A)^{-1}f$, $g_\varepsilon = (I+\varepsilon A)^{-1}g$ and $\phi_\varepsilon = (I+\varepsilon A)^{-1}\phi$. Then $\partial_t\phi_\varepsilon(t,\cdot) \in D$ and $\partial_{tt}\phi_\varepsilon(t,\cdot) \in \mathcal{E}$, with their norms uniformly bounded in t, while $\phi_\varepsilon(t,\cdot) \to \phi(t,\cdot)$ in D and $\partial_t\phi_\varepsilon(t,\cdot) \to \partial_t\phi(t,\cdot)$ in $\mathcal{E}$ when $\varepsilon \to 0$. The conclusion readily follows.

When $f \in \mathcal{E}$ and $g \in H$, we define $\phi(t,\cdot)$ by (14). Then $\phi(t,\cdot) \in \mathcal{E}$ and $\partial_t\phi(t,\cdot) \in H$. The continuity results are obtained by density arguments in the same way as above.

The reader should notice that in this case we have $\partial_{tt}\phi(t,\cdot) + A(\phi(t,\cdot)) = 0$ in $\mathcal{E}'$, where $\mathcal{E}'$ is the dual space of $\mathcal{E}$; hence ϕ is a weak solution of (P). Regarding the conservation of the energy, although the argument here is standard, we recall it for its convenience. We assume first that $f \in D$ and $g \in \mathcal{E}$. Then $\phi(t,\cdot)$ is a strong solution of (P) and we have

$$\int_{t_1}^{t_2}\int_\Omega a(z)\,\partial_t\phi\,(\partial_{tt}\phi + A\phi)\,dt\,d\mu = 0. \tag{15}$$

We consider each term separately, obtaining for the first one

$$\int_\Omega\int_{t_1}^{t_2} a(z)\,\partial_t\phi\ \partial_{tt}\phi\,dt\,d\mu = \frac{1}{2}\int_\Omega a(z)\,(\partial_t\phi)^2\,d\mu\Big|_{t_1}^{t_2}, \tag{16}$$

and for the second one (see for instance [6])

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \partial_t\phi\ A\phi\, a(z)\, dt\, d\mu &= \int_{t_1}^{t_2} < \partial_t\phi, A\phi >_H\ dt \\ &= \int_{t_1}^{t_2} b(\phi,\partial_t\phi)\, dt \\ &= \frac{1}{2}\int_\Omega \left(a(z)\,(\partial_t\phi)^2 + b(z)\,(\partial_z\phi)^2 + c(z)\,|\nabla\phi|^2 + d(z)|\phi|^2\right) d\mu\ \Bigg|_{t1}^{t_2}.\end{aligned} \tag{17}$$

Now, by (15), adding (16) and (17), we have for all $t > 0$

$$\begin{aligned} E(\phi,t) &= \frac{1}{2}\int_\Omega \left(a(z)\,(\partial_t\phi)^2 + b(z)\,(\partial_z\phi)^2 + c(z)\,|\nabla\phi|^2 + d(z)|\phi|^2\right) d\mu \\ &= \frac{1}{2}\left(\|g\|_H^2 + b(f,f)\right).\end{aligned}$$

Again, by a density argument as before, this result remains true when $f \in \mathcal{E}$ and $g \in H$. □

5. Propagation of classical scalar fields in static spherically symmetric spacetimes

We consider a $(n+2)$-dimensional static and spherically symmetric spacetime with $n \geq 1$ and metric signature $(- + \cdots +)$. Due to the required isometries the more general line element can be written as

$$ds^2 = -F(r)\, dt^2 + G(r)\, dr^2 + r^2\ d\ell^2_{S^n}\,, \tag{18}$$

where $d\ell^2_{S^n}$ is the metric on the unit n-sphere S^n and r in $(0,+\infty)$. For a non-degenerate Lorentzian metric g_{ab}, (18) makes sense only for those values of r such that $0 < F(r)G(r) < +\infty$. On the other hand, since $g_{ab}(\partial_t)^a(\partial_t)^b = -F$, the Killing vector field ∂_t is timelike only in the region $F(r) > 0$, and so spacetime is static only in this region. Therefore, without loss of generality, from now on we shall restrict ourselves to the region where $F(r)$ and $G(r)$ are both finite and positive. In addition we shall assume that F and G are such that the condition $0 < F(r), G(r) < +\infty$ holds in a finite union of disjoint non empty open subintervals (r_i^-, r_i^+) of $(0,+\infty)$ and $F, F', G \in C^1(\bigcup_{i=1}^m (r_i^-, r_i^+))$. If the spacetime is *asymptotically flat*, in the outer region $(r_m^-, +\infty)$, we can find coordinates such that $\lim_{r\to+\infty} F(r) = \lim_{r\to+\infty} G(r) = 1$.

Due to the required symmetries the more general energy-momentum tensor can be written as

$$T_a^b = \mathrm{diag}\{-\rho(r), p_r(r), p_\theta(r), \ldots, p_\theta(r)\}\,, \tag{19}$$

where $\rho(r)$ is the energy density, and $p_r(r)$, $p_\theta(r)$ are the principal pressures. We shall assume that $\rho(r)$ is bounded and the *dominant energy condition*[4] is satisfied, which, in this case, is equivalent to

$$|p_r(r)|, |p_\theta(r)| \leq \rho(r) < +\infty\,. \tag{20}$$

From (18) and (19) we get that Einstein's equations, i.e., $G_{ab} + \Lambda\, g_{ab} = 8\pi T_{ab}$, become

$$G_t^t = -\frac{n}{2\,r^2}\left((n-1)\left(1 - \frac{1}{G(r)}\right) + r\,\frac{G'(r)}{G(r)^2}\right) = -8\pi\,\rho(r) - \Lambda\,, \tag{21}$$

$$G_r^r = \frac{n}{2\,r^2}\left(\frac{r\,F'(r)}{F(r)G(r)} + (n-1)\left(\frac{1}{G(r)} - 1\right)\right) = 8\pi\, p_r(r) - \Lambda\,, \tag{22}$$

[4]See for example [7]

$$G_\theta^\theta = \frac{F''(r)}{2F(r)G(r)} - \frac{F'(r)G'(r)}{4F(r)G(r)^2} + \frac{(n-1)F'(r)}{2rF(r)G(r)} - \frac{F'(r)^2}{4F(r)^2G(r)}$$
$$-\frac{(n-1)G'(r)}{2rG(r)^2} - \frac{(n-2)(n-1)}{2r^2}\left(1 - \frac{1}{G(r)}\right) = 8\pi\, p_\theta(r) - \Lambda\,, \tag{23}$$

where Λ is the cosmological constant. Furthermore, the local energy-momentum conservation ($\nabla_a T^{ab} = 0$) gives

$$p_r'(r) = -\frac{\rho(r) + p_r(r)}{2}\,\frac{F'(r)}{F(r)} - n\,\frac{(p_r(r) - p_\theta(r))}{r}\,. \tag{24}$$

Of course, due to Bianchi's identities, (21)-(24) are not independent. These are a system of three linear independent ODE's and, in order to find the five unknown functions $F(r)$, $G(r)$, $\rho(r)$, $p_r(r)$ and $p_\theta(r)$, we have to provide *equations of state* relating the functions $\rho(r)$, $p_r(r)$ and $p_\theta(r)$.

From (21) and (22) we can write down a more handleable set of two equivalent independent equations

$$\left(r^{n-1}\left(1 - \frac{1}{G(r)}\right)\right)' = \frac{2r^n}{n}\big(8\pi\,\rho(r) + \Lambda\big)\,, \tag{25}$$

$$\ln'\Big(F(r)\,G(r)\Big) = \frac{16\pi}{n}\Big(\rho(r) + p_r(r)\Big)\,r\,G(r)\,, \tag{26}$$

which in the vacuum cases, leads readily to the solution.

Indeed, if we for instance set $\rho(r) = -p_r(r) = p_\theta(r)$, from (24) we immediately get that

$$p_r(r) = -\rho(r) = -\frac{C_1}{r^{2n}}\,,$$

where the constant C_1 must be positive by (20). Then, we find from (25) that

$$\frac{1}{G(r)} = 1 - \frac{C_2}{r^{n-1}} + \frac{16\pi\; C_1}{n(n-1)\,r^{2n-2}} - \frac{2\,\Lambda\,r^2}{n(n+1)}\,,$$

where C_2 is a new arbitrary constant. And (26) immediately gives $F(r)\,G(r) = C_3$, and we can always set the constant $C_3 = 1$ by scaling the time. This family of solutions, depending on three parameters, includes the higher-dimensional generalization of Schwarzschild, de Sitter and Reissner-Nordström geometries.

For future use, we shall prove the following result.

Lemma 5.1. *If* $0 < F(r), G(r) < +\infty$ *in some interval* (r_i^-, r_i^+)*, then*

(1) $F(r)G(r)$ *is a nondecreasing function of* r *in* (r_i^-, r_i^+)*, and then bounded in a neighborhood of* r_i^-*.*
(2) *In the outer region of an asymptotically flat spacetime,* $F(r)G(r)$ *is bounded.*

Proof:

(i) As a consequence of the dominant energy condition (20) the right hand side of (26) cannot be negative, then $F(r)G(r)$ cannot be decreasing.

(ii) Since $F(r)G(r)$ is nondecreasing, we get that $0 < F(r)\,G(r) \le 1$ since $\lim_{r\to+\infty} F(r) = \lim_{r\to+\infty} G(r) = 1$.

□

In these spacetimes, we shall consider the propagation of a scalar field ψ with Lagrangian density

$$\mathscr{L} = -\frac{1}{2}\nabla^a\psi\,\nabla_a\psi - \frac{m^2}{2}\,\psi^2,$$

where the constant m is the mass of the field and ∇ denotes the covariant derivative (Levi-Civita connection).

As usual, we obtain the field equations by requiring that the action

$$S = \int \mathscr{L}(\nabla_a \psi, \psi, g_{ab}) \sqrt{|g|}\; dt d\mu$$

be stationary under arbitrary variations of the fields $\delta\psi$ in the interior of any compact region, but vanishing at its boundary. Thus, we have the Euler-Lagrange equation

$$\nabla_a \left(\frac{\partial \mathscr{L}}{\partial \nabla_a \psi} \right) = \frac{\partial \mathscr{L}}{\partial \psi} \;,$$

which, in our case, becomes the Klein-Gordon equation

$$\nabla_a \nabla^a \psi \; = \Box \psi = \frac{\partial_a \left(\sqrt{|g|}\; g^{ab}\, \partial_b \psi \right)}{\sqrt{|g|}} = m^2\, \psi \,. \tag{27}$$

Therefore, we get from (18) and (27) that the field equation may be written as

$$\partial_{tt} \psi = -\, A\psi$$

where

$$A\psi = -\frac{1}{r^n} \sqrt{\frac{F(r)}{G(r)}} \left(\partial_r \left(r^n \sqrt{\frac{F(r)}{G(r)}}\, \partial_r \psi \right) + r^{n-2}\; \sqrt{F(r) G(r)}\; \Delta_{S^n} \psi \right) + m^2\, F(r)\, \psi \;, \tag{28}$$

where Δ_{S^n} is the Laplacian on the unit n-sphere. Then, by comparing with the operator defined in (2), we get the identification of the coefficients

$$\begin{aligned} a(r) &= r^n \sqrt{\tfrac{G(r)}{F(r)}} \;, & b(r) &= r^n \sqrt{\frac{F(r)}{G(r)}} \;, \\ c(r) &= r^{n-2} \sqrt{F(r) G(r)} \;, & d(r) &= m^2\, r^n \sqrt{F(r) G(r)} \;. \end{aligned} \tag{29}$$

Remark 5.2. From (18) we get that radial null geodesics satisfy $\dfrac{dt}{dr} = \pm \sqrt{\dfrac{G(r)}{F(r)}}$. Then, if r_0 and r belong to the closure of a connected region where $0 < F(s), G(s) < +\infty$, we find from (29) that the coordinate time t a radial photon takes to travel from r to r_0 is

$$T(r \to r_0) = \left| \int_r^{r_0} \sqrt{\frac{G(s)}{F(s)}}\; ds \right| = \left| \int_r^{r_0} \sqrt{\frac{a(s)}{b(s)}}\; ds \right| \,. \tag{30}$$

We shall see that it is actually this time which plays a crucial role in the analysis of *e.s.a.* when there is a horizon at r_0 ($r_0 = r_i^+$ or $r_0 = r_i^-$) in the spacetime, i.e., $T(r \to r_0) = +\infty$.

Lemma 5.3. *In the outer region of an asymptotically flat spacetime one has* $\displaystyle\int^{+\infty} a(r)\, dr = +\infty$.

Proof. If $\displaystyle\lim_{r\to+\infty} F(r) = \lim_{r\to+\infty} G(r) = 1$ by (29) we have that $\displaystyle\lim_{r\to+\infty} \frac{a(r)}{r^n} = 1$, and then $\displaystyle\int^{+\infty} a(r)\, dr = +\infty$. $\square$

Lemma 5.4. *If $0 < F(r), G(r) < +\infty$ in (r_i^-, r_i^+), with $r_i^- > 0$, the three following statements are equivalent*

$$\int_{r_i^-} \frac{1}{b(r)}\, dr = +\infty, \qquad \int_{r_i^-} a(r)\, dr = +\infty \quad \text{and} \quad \int_{r_i^-} \sqrt{\frac{a(r)}{b(r)}}\, dr = +\infty \,.$$

On the other hand, if r_i^+ is finite, the three following statements are equivalent

$$\int^{r_i^+} \frac{1}{b(r)}\,dr = +\infty, \qquad \int^{r_i^+} a(r)\,dr = +\infty \quad \text{and} \quad \int^{r_i^+} \sqrt{\frac{a(r)}{b(r)}}\,dr = +\infty\,.$$

Proof.

By (29) we have that $a(r)b(r) = r^{2n}$. For $r_* < r < r^* < +\infty$, we readily get the inequalities

$$\frac{r_*^{2n}}{b(r)} < a(r) < \frac{r^{*\,2n}}{b(r)} \qquad \text{and} \qquad r_*^n \sqrt{\frac{a(r)}{b(r)}} < a(r) < r^{*\,n} \sqrt{\frac{a(r)}{b(r)}}\,.$$

Now, by integrating these expressions between r_* and r^*, we get the result. □

Observe that by the properties of the functions F and G, under the hypotheses of lemma 5.4 we have

- $a,\, b,\, c,\, d \in C^1\left((r_i^-, r_i^+)\right)$
- $a,\, b,\, c > 0$ and $d \geq 0$ in (r_i^-, r_i^+)
- $a^{-1},\, b^{-1},\, c^{-1} \in L^1_{loc}\left((r_i^-, r_i^+)\right)$.

Then, if we consider the operator defined by (28) in $\Omega = (r_i^-, r_i^+) \times S^n$, we have:

Theorem 5.5. *Let A be the operator corresponding to the propagation of a scalar field in $\Omega = (r_m^-, \infty)\times S^n$ in a static, spherically symmetric and asymptotically flat spacetime where the dominant energy condition holds. The three following statements are equivalent:*

1. *The time $T(r \to r_m^-)$ is infinite.*
2. *A is a q.e.s.a. operator.*
3. *A is an e.s.a. operator.*

Or, in other words, A is e.s.a. if and only if a radial photon needs an infinite amount of time to get r_m^-.

Proof:

(i) ⇒ *(ii)* and *(iii)*: By lemma 5.3 we have that $\int^{+\infty} a(r)\,dr = +\infty$. On the other hand, if $T(r \to r_m^-) = +\infty$, it follows by (30) that $\int_{r_m^-} \sqrt{\frac{a(r)}{b(r)}}\,dr = +\infty$, and then from lemma 5.4 we have $\int_{r_m^-} a(z)\,dz = +\infty$. Therefore, it follows from theorem 3.24 that the operator A is *q.e.s.a* and from theorem 3.25 *(i)* that the operator A is *e.s.a*.

(ii) ⇒ *(i)*: Conversely, assume that $T(r \to r_m^-) < +\infty$, then $\int_{r_m^-} \sqrt{\frac{a(r)}{b(r)}}\,dr < +\infty$. And it immediately follows from lemma 5.4 that $\int_{r_m^-} a(r)\,dr < +\infty$ and $\int_{r_m^-} \frac{1}{b(r)}\,dr < +\infty$. On the other hand, since $F(r)G(r)$ is bounded by lemma 5.1, $\int_{r_m^-} d(r)\,dr = m^2 \int_{r_m^-} r^n \sqrt{F(r)G(r)}\,dr < +\infty$. Therefore, it follows from theorem 3.24 that the operator A is not *q.e.s.a*.

(iii) ⇒ *(ii)*: This is obvious by definition. □

Remark 5.6. Note that the boundedness of $F(r)G(r)$ is only used in the proof of the sufficiency of the condition $T(r \to r_m^-) = +\infty$, to guarantee that $d(r)$ is integrable at r_m^-. Therefore, for massless fields, since in this case $d(r) \equiv 0$ the theorem follows without invoking any energy condition.

Similar results also follow from remark 3.23 and lemma 5.4 at internal horizons.

6. Examples

6.1. (n+2)**-dimensional punctured Minkowski spacetime.** Here we consider the flat (n+2)-dimensional Minkowski spacetime with a removed spatial point. We chose the origin of coordinates at this point and then the line element can be written as

$$ds^2 = -dt^2 + dr^2 + r^2\; dl^2_{S^n}\ ,$$

where $-\infty < t < +\infty$ and $0 < r < +\infty$. This spacetime has a time-like singular boundary along the t axis. In this case, $\Omega = (0,\infty) \times S^n$ and $F(r) = G(r) = 1$, so the coefficients in (29) are $a(r) = b(r) = r^n$, $c(r) = r^{n-2}$ and $d(r) = m^2\, r^n$. The operator A in (28) turns out to be

$$A\psi = -\frac{1}{r^n}\;\partial_r\,(r^n\,\partial_r\psi) - \frac{1}{r^2}\,\Delta_{S^n}\psi + m^2\,\psi\ ,$$

which formally is nothing but $-\Delta + m^2$.

Now, for $n \geq 1$, we have that $\int^{+\infty} a(r)\,dr = +\infty$ and $\int_0 \frac{dr}{b(r)} = +\infty$. Then it immediately follows from theorem 3.2 that A is a *q.e.s.a.* operator for every $m^2 \geq 0$ and every $n \geq 1$.

We turn now to explore whether A is an *e.s.a.* operator too. Taking into account that $d(z)/a(z) = m^2$, $\int_0 a(z)\,dz = \int_0 r^n\,dz < +\infty$ and $\int^{+\infty} a(z)\,dz = +\infty$, we can apply corollary 3.22.

Now, for $0 < r_1 < +\infty$, we have

$$\beta_0(r) = \int_r^{r_1} \frac{du}{b(u)} = \begin{cases} -\ln\left(\frac{r}{r_1}\right) & if n = 1 \\ \frac{r^{1-n} - {r_1}^{1-n}}{n-1} & \text{if} n \geq 2\ . \end{cases}$$

Thus,

$$\int_0^{r_1} \beta_0^2(r) a(r) dr < +\infty$$

if and only if $n = 1, 2$. Therefore, it immediately follows from corollary 3.22 that A is an *e.s.a.* operator only if $n \geq 3$. This is a well known result, see for instance [8, 9].

6.2. $(n+2)$**-dimensional anti-Schwarzschild** $(M < 0)$ **spacetime.** Here we consider the $(n+2)$-dimensional spacetime with line element

$$ds^2 = -\left(1 + \frac{r_s^{n-1}}{r^{n-1}}\right) dt^2 + \left(1 + \frac{r_s^{n-1}}{r^{n-1}}\right)^{-1} dr^2 + r^2\; d\Omega^2_{S^n}\ ,$$

where $-\infty < t < +\infty$, $0 < r < +\infty$, r_s is a positive constant and $n \geq 2$[5]. This spacetime has a naked timelike singularity at $r = 0$ where some componentes of the Weyl tensor diverge.

In this case, $\Omega = (0,\infty) \times S^n$ and we get from (29) that the coefficients of the operator A are

$$a(r) = \frac{r^{2n-1}}{r^{n-1} + r_s^{n-1}} \quad , \quad b(r) = r(r^{n-1} + r_s^{n-1}) \quad , \quad c(r) = r^n \quad \text{and} \quad d(r) = m^2\, r^n\ .$$

We get therefore

$$\int_0 \frac{dr}{b(r)} = +\infty \quad \text{and} \quad \int^{+\infty} a(r)\,dr = +\infty.$$

Then it immediately follows from theorem 3.2 that A is a *q.e.s.a.* operator for every $m^2 \geq 0$ and every $n \geq 2$.

For $m = 0$ and $n = 2$, we have already proved in [1] that A is not an *e.s.a.* operator. Here, we shall analyze the general case.

[5] The case $n = 1$ is 3-dimensional Minkowski spacetime already discussed in 6.1

We first consider the case $m = 0$. Taking into account that

$$\int_0 a(r)\,dr < +\infty\,, \quad \int^{+\infty} a(r)\,dr = +\infty \text{ and } \; d(z) = 0,$$

we can apply corollary 3.22.

For $0 < r < r_s$, we have

$$\beta_0(r) = \int_r^{r_s} \frac{ds}{b(s)} = \frac{-1}{r_s^{n-1}(n-1)} \ln\left(\frac{2\,r^{n-1}}{r^{n-1}+r_s^{n-1}}\right)\,,$$

and

$$\int_0^{r_s} \beta_0^2(r)a(r)dr < +\infty\,.$$

Thus, in the massless case, A is not an *e.s.a.* operator for every $n \geq 2$ thanks to the corollary 3.22.

For $m^2 > 0$ we cannot apply corollary 3.22 since $d(z)/a(z)$ is not bounded near 0. Nevertheless, the ordinary differential equation (8), satisfied by the function $\alpha(z)$ of lemma 3.13, becomes in this case

$$-\Big(r\,(r^{n-1}+r_s^{n-1})\,\alpha'(r)\Big)' + m^2\,r^n\,\alpha(r) = 0\,,$$

and a straightforward computation shows that

$$\alpha(z) = \alpha(0)\left(1 + \frac{m^2\,r_s^2}{(n+1)^2}\,\left(\frac{r}{r_s}\right)^{n+1} - \frac{m^2\,r_s^2}{2n(n+1)}\,\left(\frac{r}{r_s}\right)^{2n} + \ldots\right)$$

near 0. Furthermore, since by lemma 3.13 $\alpha(r)$ is positive and increasing in $(0, r_s)$, and by definition $\alpha(r_s) = 1$, we get that $0 < \alpha(0) < 1$.

Therefore

$$\beta(r) = \alpha(r)\int_r^{r_s} \frac{ds}{b(s)\alpha(s)^2} < \frac{1}{\alpha(r)}\int_r^{r_s}\frac{ds}{b(s)} < \frac{1}{\alpha(0)}\,\beta_0(r)$$

and

$$\int_0^{r_s} \beta^2(r)a(r)dr < \frac{1}{\alpha(0)^2}\int_0^{r_s}\beta_0^2(r)a(r)dr < +\infty\,.$$

It follows from theorem 3.9 *(ii)* that A is not an *e.s.a.* operator for every $n \geq 2$ and $m^2 \geq 0$.

Remark 6.1. *Note that the estimate*

$$\beta(z) = \alpha(z)\int_z^1 \frac{ds}{b(s)\alpha(s)^2} < \frac{1}{\alpha(z)}\,\beta_0(z)\,,$$

when $\alpha(0) \neq 0$, also gives a necessary and sufficient condition for e.s.a. in terms of $\beta_0(z)$ only.

For analytic $b(z)$ and $d(z)$, as in our example, $\alpha(0) \neq 0$ if one of the roots of the indicial polynomial of (8) is zero and the other non positive, which requires that

$$\lim_{z\to 0^+} \frac{z^2 d(z)}{b(z)} = 0 \qquad \text{and} \qquad \lim_{z\to 0^+}\frac{z\,b'(z)}{b(z)} \geq 1\,.$$

6.3. $(n+2)$**-dimensional Schwarzschild-Tangherlini spacetime.** Here we consider the $(n+2)$-dimensional spacetime with line element

$$ds^2 = -\left(1 - \frac{r_s^{n-1}}{r^{n-1}}\right)dt^2 + \left(1 - \frac{r_s^{n-1}}{r^{n-1}}\right)^{-1} dr^2 + r^2\; d\Omega^2_{S^n}\,,$$

where r_s is a positive constant, $-\infty < t < +\infty$, $0 < r < r_s$ or $r_s < r < +\infty$ and $n \geq 2$. This spacetime has a spacelike irremovable singularity at $r = 0$ where some components of the Riemann tensor diverge and an event horizon at $r = r_s$, the latter may be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension [10]. As already mentioned, our wave formulation only

makes sense in the static region ($r_s < r < +\infty$), and we will use it to explore the properties of the wave equation (27) in this region.

Thus, we consider the operator A given by (28) in $\Omega = (r_s, \infty) \times S^n$, and we see from (29) that

$$a(r) = \frac{r^{2n-1}}{r^{n-1} - r_s^{n-1}} \quad , \quad b(r) = r(r^{n-1} - r_s^{n-1}) \quad \text{and} \quad d(r) = m^2\, r^n \ .$$

Now, we get from (30) that

$$T(r \to r_s) = \int_{r_s}^{r} \left(\frac{a(s)}{b(s)}\right)^{\frac{1}{2}} ds = \int_{r_s}^{r} \frac{s^{n-1}}{s^{n-1} - r_s^{n-1}}\, ds = +\infty \ .$$

Therefore, it immediately follows from theorem 5.5 that A is an *e.s.a.* operator in $\Omega = (r_s, \infty) \times S^n$ for every $n \geq 2$ and any $m^2 \geq 0$, and the Cauchy problem is well-posed without requiring any boundary condition at the event horizon.

6.4. $(n+2)$**-dimensional Reissner-Nordström spacetime.** Here we consider the $(n+2)$-dimensional spacetime with line element

$$ds^2 = -\left(1 - \frac{r_s^{n-1}}{r^{n-1}} + \frac{q^{2n-2}}{4\, r^{2n-2}}\right) dt^2 + \left(1 - \frac{r_s^{n-1}}{r^{n-1}} + \frac{q^{2n-2}}{4\, r^{2n-2}}\right)^{-1} dr^2 + r^2\ d\Omega^2_{S^n} \ ,$$

where r_s and q^2 are positive constants and $n \geq 2$[6]. If $q^2 > r_s^2$ the metric is non-singular everywhere except for the timelike irremovable repulsive singularity at $r = 0$. If $q^2 \leq r_s^2$, the metric also has singularities at r_+ and r_-, where $r_\pm^{n-1} = (r_s^{n-1} \pm \sqrt{r_s^{2n-2} - q^{2n-2}})/2$; it is regular in the regions defined by $\infty > r > r_+$, $r_+ > r > r_-$ and $r_- > r > 0$ (if $q^2 = r_s^2$ only the first and the third regions exist). As in the Schwarzschild case, these singularities may be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension [11, 12]. The first and the third regions are both static, whereas the second region (when it exists) is spatially homogeneous but not static.

We shall study the properties of the wave equation in the static regions. For convenience we shall analyze separately the three cases. Note that, in the three cases this spacetime is asymptotically flat.

6.4.1. *Case* $q^2 > r_s^2$. This spacetime has only a naked timelike irremovable repulsive singularity at $r = 0$. In this case, we consider the operator A given by (28) in $\Omega = (0, \infty) \times S^n$, and from (29) we have

$$a(r) = \frac{r^n}{1 - \dfrac{r_s^{n-1}}{r^{n-1}} + \dfrac{q^{2n-2}}{4\, r^{2n-2}}} \ , \quad b(r) = r^n - r_s^{n-1} r + \frac{q^{2n-2}}{4\, r^{n-2}} \quad \text{and} \quad d(r) = m^2 r^n \ .$$

Then

$$\int_0 \frac{dr}{b(r)} + a(r) + d(r)\, dr < +\infty \, .$$

Hence it follows from theorem 3.2 *(ii)* that A is not even a *q.e.s.a.* operator in this case, for every $n \geq 2$ and any $m^2 \geq 0$. Therefore, in contrast to the anti-Schwarzschild case, in order to have a well-possed Cauchy problem a boundary condition at the singularity must be given.

[6] The case $n = 1$ is again 3-dimensional Minkowski spacetime already discussed in 6.1

6.4.2. *Case $r_s^2 = q^2$ (extreme case).* This spacetime also has a removable singularity at $r_* = 2^{\frac{-1}{n-1}} r_s$. In this case, we consider the operator A given by (28) in two regions $\Omega = (0, r_*) \times S^n$ or $\Omega = (r_*, \infty) \times S^n$.

We get from (29) that

$$a(r) = \frac{r^{3n-2}}{\left(r^{n-1} - r_*^{n-1}\right)^2} \quad , \quad b(r) = \frac{\left(r^{n-1} - r_*^{n-1}\right)^2}{r^{n-2}} \quad \text{and} \quad d(r) = m^2\, r^n \ .$$

We first consider the outer region ($r_* < r < +\infty$). In this case, we get from (30) that

$$T(r \to r_*) = \int_{r_*}^{r} \left(\frac{a(s)}{b(s)}\right)^{\frac{1}{2}} \, ds = \int_{r_*}^{r} \frac{s^{2n-2}}{\left(s^{n-1} - r_*^{n-1}\right)^2} \, ds = +\infty \ .$$

Therefore, it follows from theorem 5.5 that A is an *e.s.a.* operator in $\Omega = (r_*, \infty) \times S^n$ for every $n \geq 2$ and any $m^2 \geq 0$, and the Cauchy problem is well-posed without requiring any boundary condition at the event horizon.

Regarding the inner region $0 < r < r_*$, we get that

$$\int_0 \left(\frac{1}{b(z)} + d(z) + a(z)\right) \, dz < +\infty \, .$$

Hence it follows from theorem 3.24 that A is not even a *q.e.s.a.* operator, for every $n \geq 2$ and any $m^2 \geq 0$.

However, we have

$$\int^{r_*} a(r) \, dr = \int^{r_*} \frac{r^{3n-2}}{\left(r^{n-1} - r_*^{n-1}\right)^2} \, dr = +\infty \, ,$$

so it follows from remark 3.23 that in order to have a well-posed Cauchy problem in $\Omega = (0, r_*) \times S^n$ a boundary condition at the singularity ($r = 0$) must be given but not at the horizon ($r = r_*$).

6.4.3. *Case $r_s^2 > q^2$.* This spacetime has, besides the timelike irremovable repulsive singularity at $r = 0$, two removable singularities at r_+ and r_-. In this case, we consider the operator A given by (28) in two regions $\Omega = (0, r_-) \times S^n$ or $\Omega = (r_+, \infty) \times S^n$, by abuse of notation we call A these two different operators.

From (29) we can write

$$a(r) = \frac{r^{3n-2}}{\left(r^{n-1} - r_-^{n-1}\right)\left(r^{n-1} - r_+^{n-1}\right)} \ , \qquad b(r) = \frac{\left(r^{n-1} - r_-^{n-1}\right)\left(r^{n-1} - r_+^{n-1}\right)}{r^{n-2}}$$

$$\text{and} \qquad d(r) = m^2\, r^n \ .$$

We first consider the outer region ($r_+ < r < +\infty$). In this case, we get from (30) that

$$T(r \to r_*) = \int_{r_*}^{r} \left(\frac{a(s)}{b(s)}\right)^{\frac{1}{2}} \, ds = \int_{r_+}^{r} \frac{s^{2n-2}}{\left(s^{n-1} - r_-^{n-1}\right)\left(s^{n-1} - r_+^{n-1}\right)} \, ds = +\infty \ .$$

Therefore, it follows from theorem 5.5 that A is an *e.s.a.* operator in $\Omega = (r_+, \infty) \times S^n$ for every $n \geq 2$ and any $m^2 \geq 0$, and the Cauchy problem is well-posed without requiring any boundary condition at the event horizon.

Regarding the inner region $0 < r < r_-$, we get

$$\int_0 \left(\frac{1}{b(z)} + d(z) + a(z)\right) \, dz < +\infty \, .$$

Hence it follows from theorem 3.24 that A is not even a *q.e.s.a.* operator, for every $n \geq 2$ and any $m^2 \geq 0$.

However, we have

$$\int^{r_*} a(r) \, dr = \int^{r_*} \frac{r^{3n-2}}{\left(r_-^{n-1} - r^{n-1}\right)\left(r_+^{n-1} - r^{n-1}\right)} \, dr = +\infty \, ,$$

so it follows from remark 3.23 that in order to have a well-posed Cauchy problem in $\Omega = (0, r_-) \times S^n$ a boundary condition at the singularity ($r = 0$) must be given but not at the horizon ($r = r_-$).

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